

Two-dimensional Euler equations with gravity

Ralf Deiterding

Institute of Mathematics, Technical University Cottbus, Germany

e-mail: deiterding@math.tu-cottbus.de

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1 Introduction

Instationary flow phenomena of gases are described by the Euler equations. While various high-resolution methods have been developed to solve the homogenous Euler equations numerically, the appropriate incorporation of source terms is still a topic of current research.

In this report, we calculate the flow of a single ideal gas under influence of the standard gravitational field. In this particular case, a source term modifying the equations of momentum and energy has to be considered.

A smooth two-dimensional advection example is chosen for which the exact solution is known. Error norms can easily be calculated allowing the validation and rating of different numerical methods. Accurate numerical results can be recomputed with minimal effort by standard fractional step methods and can serve as a standard when testing new numerical methods.

2 Governing equations

The two-dimensional Euler equations with standard gravity in y -direction take the following form:

$$\begin{aligned}
 \rho_t &+ (\rho u)_x &+ (\rho v)_y &= 0 \\
 (\rho u)_t &+ (\rho u^2 + p)_x &+ (\rho uv)_y &= 0 \\
 (\rho v)_t &+ (\rho uv)_x &+ (\rho v^2 + p)_y &= -\rho g \\
 (\rho E)_t &+ [u(\rho E + p)]_x &+ [v(\rho E + p)]_y &= -\rho v g
 \end{aligned} \tag{1}$$

with the equation of state

$$p = (\gamma - 1) \left[\rho E - \rho \left(\frac{u^2}{2} + \frac{v^2}{2} \right) \right]$$

Here $\rho = \rho(x, y, t)$ is the density, $p = p(x, y, t)$ is the hydrodynamic pressure and $E = E(x, y, t)$ is the total energy per unit mass. $u = u(x, y, t)$ is the velocity in x -direction, while $v = v(x, y, t)$ is the velocity in y -direction. g is the gravitational constant acting in y -direction. γ is the constant adiabatic coefficient for the gas.

The system of equations (1) reduces to a single transport equation for the density ρ for

$$\begin{aligned}
 u(x, y, t) &= u_0 \\
 v(x, y, t) &= v_0 - gt \\
 p(x, y, t) &= p_0
 \end{aligned} \tag{2}$$

with $u_0, v_0 \in \mathbb{R}$; $p_0 \in \mathbb{R}^+$. In this case, an arbitrary initial density distribution $\rho_0 \in C^k(\mathbb{R}^2)$ is simply transported:

$$\rho(x, y, t) = \rho_0(x - u_0t, y - v_0t + \frac{1}{2}gt^2) \quad (3)$$

3 Advection of a smooth density distribution

To measure the rate of convergence of a numerical method on smooth solutions we consider an advection example with $\rho_0 \in C^1(\mathbb{R}^2)$.

3.1 Domain and boundary conditions

We use the rectangular domain

$$\Omega = \{(x, y) \in \mathbb{R}^2; x, y \in [0, 2]\}$$

with periodic boundary conditions on both sides.

3.2 Initial conditions and exact solution

We employ initial conditions on Ω for pressure and velocities according to (2). For ρ_0 we use a smooth density distribution radial symmetric around $C = (x_0, y_0)$. The choice of periodic boundary conditions ensures that no disturbances in pressure and velocity can emerge from the boundaries and the initial density distribution is simply advected, while the equations of motion for the center C are

$$\left(x_c(t) = x_0 + u_0t, y_c(t) = y_0 + v_0t - \frac{1}{2}gt^2 \right) . \quad (4)$$

For the actual calculations we use the density distribution

$$r(x, y, t) = \sqrt{(x - x_c(t))^2 + (y - y_c(t))^2} \quad (5)$$

$$\rho_r(r) = \begin{cases} \rho_a + \rho_b \left[\sin\left(\frac{r}{R}\pi + \frac{\pi}{2}\right) + 1 \right] & \text{if } 0 \leq r < R \\ \rho_a & \text{if } R \leq r \end{cases} \quad (6)$$

with $\rho(x, y, t) = \rho_r(r(x, y, t))$, $\rho_a = 1.0$, $\rho_b = 0.05$.

The remaining parameters are:

$$x_0 = 0.75, y_0 = 0.75, R = 0.5, u_0 = 1, v_0 = 1.25, p_0 = 1, g = 1, \gamma = 1.39$$

3.3 Calculations

We utilize the wave propagation method of R. J. LeVeque [2] and incorporate the source term via a fractional step method. These methods alternate between solving the homogenous conservation law and an ordinary differential equation for the source term.

For the actual calculations the popular fractional step method of Strang [4] is used. The ordinary differential equation of the source term is solved with a two-step Runge-Kutta method. A detailed explanation of the algorithm can be found in [1].

As our example is smooth and the source term is nonstiff Strang splitting should be an appropriate choice for a numerical method [3]. The separate steps of the fractional step method are of second order (the MC-limiter is used for the transport step) and the whole numerical method can be expected to give second order accurate results on sufficiently fine grids.

Four calculations on equally spaced cartesian grids with mesh widths $h = 0.05, 0.025, 0.0125$ and 0.00625 are carried out to measure the accuracy of the numerical method. The fixed time step of each calculation is $k = h/4$ yielding a CFL-No. around 0.58. The computations end at $t = 0.5$, when the center C crosses the diagonal at $(1.25, 1.25)$ again (see fig.1).

We use the L_1 -Norm of the error between exact solution f and computed solution F

$$\|E_f\|_1 = \int_{\Omega} |F - f| d\Omega$$

to compute the global error of density ρ and velocity vector \vec{v} .

While the computed pressure and velocity components vary against the exact solution in a magnitude of the machine precision, the global error for the density distribution clearly shows the expected rate of convergence (see tab. 1).

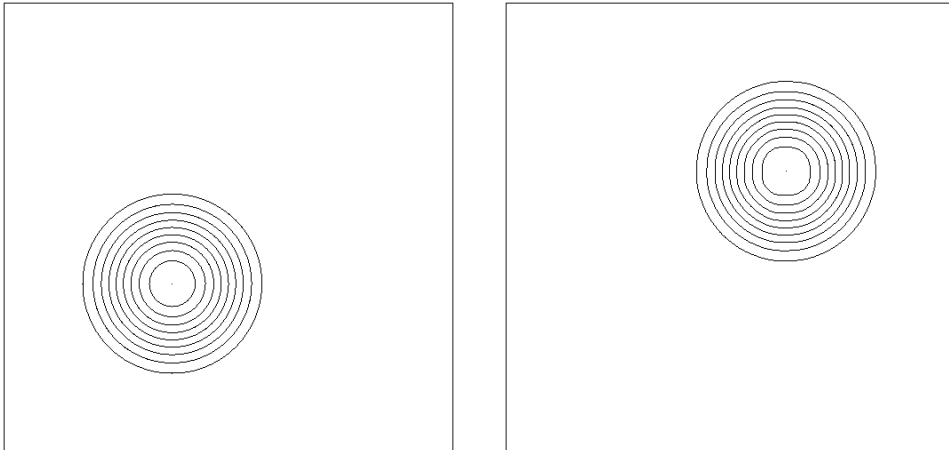


Figure 1: Contour plot of density at $t = 0$ (left) and $t = 0.5$ (right) on a 80×80 grid.

Grid cells	h	$\ E_\rho\ _1$	EOC	$\ E_{\vec{v}}\ _1$
40×40	0.05	6.06087e-04		5.59049e-15
80×80	0.025	1.66710e-04	1.86	1.93635e-14
160×160	0.0125	0.44816e-04	1.90	3.52313e-14
320×320	0.00625	0.114182e-04	1.97	3.03246e-14

Table 1: L_1 -Norms of global errors

3.4 Required output

Beside a detailed description of the algorithm (especially treatment of the source term, handling of boundary conditions, etc.), the following output is required to rate a new numerical method:

1. One-dimensional plot of ρ , u , v and p along the diagonal from lower left to upper right corner at times $t = 0, t = 0.5$ on a 80×80 grid.
2. Two-dimensional contour plot of ρ at times $t = 0, t = 0.5$ on a 80×80 grid. Contour plots of u , v of p should only be generated, if significant differences against the exact solution occur.
3. Calculation of L_1 -norms and order of convergence like in tab. 1.

References

- [1] R. J. LeVeque. CLAWPACK *software*. Available on the Web at <http://www.amath.washington.edu/~rjl/clawpack.html>.
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