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A Lattice Boltzmann Method on structured non-Cartesian meshes

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Outline

- Lattice Boltzmann methods
  - Cartesian standard scheme
  - LBM in curvilinear coordinates
  - Scheme construction
- Results
  - 2D lid-driven cavity (stretched and skewed)
  - 2D circular cylinder and comparison with AMROC-LBM
  - 2D NACA0012 airfoil
- Conclusions

Standard lattice Boltzmann scheme

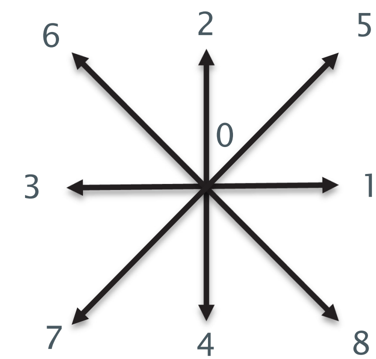
Based on solving the Boltzmann equation with the simplified BGK collision operator.

$$\frac{\partial f_\alpha}{\partial t} + \boldsymbol{e_\alpha} \cdot \nabla f_{\boldsymbol{\alpha}} = -\frac{1}{\tau}\left(f_\alpha - f_\alpha^{eq}\right), \qquad for\ \ \alpha = 0,1,2 \ldots N$$

Discrete velocities

$$\boldsymbol{e_\alpha} = \left\{ \begin{array}{cc} 0, & \alpha = 0 \\ (\pm 1,0)c, (0, \pm 1)c & \alpha = 1,3,2,4 \\ (\pm 1, \pm 1)c & \alpha = 5,6,7,8 \end{array} \right\}$$

$$c = \frac{\Delta x}{\Delta t}, \qquad Speed\ of\ sound\ c_s = \frac{c}{\sqrt{3}}$$



D2Q9 Model

Usually a two-step procedure

– Streaming $\qquad f_\alpha^*(x + \Delta t \cdot \boldsymbol{e_\alpha}, t + \Delta t) = f_\alpha(x, t)$

– Collision $\qquad f_\alpha(\cdot, t + \Delta t) = f_\alpha^*(\cdot, t + \Delta t) - \frac{1}{\tau}\left(f_\alpha^*(\cdot, t + \Delta t) - f_\alpha^{*eq}(\cdot, t + \Delta t)\right)$

Standard lattice Boltzmann scheme

Equilibrium distribution function

$$f_\alpha^{eq} = \rho w_\alpha \left[1 + \frac{3\boldsymbol{e_\alpha u}}{c^2} + \frac{9(\boldsymbol{e_\alpha u})^2}{c^2} - \frac{3\boldsymbol{u}^2}{c^2}\right]$$

$$w_0 = \frac{4}{9}, w_{1,2,3,4} = \frac{1}{9}, w_{5,6,7,8} = \frac{1}{36}$$

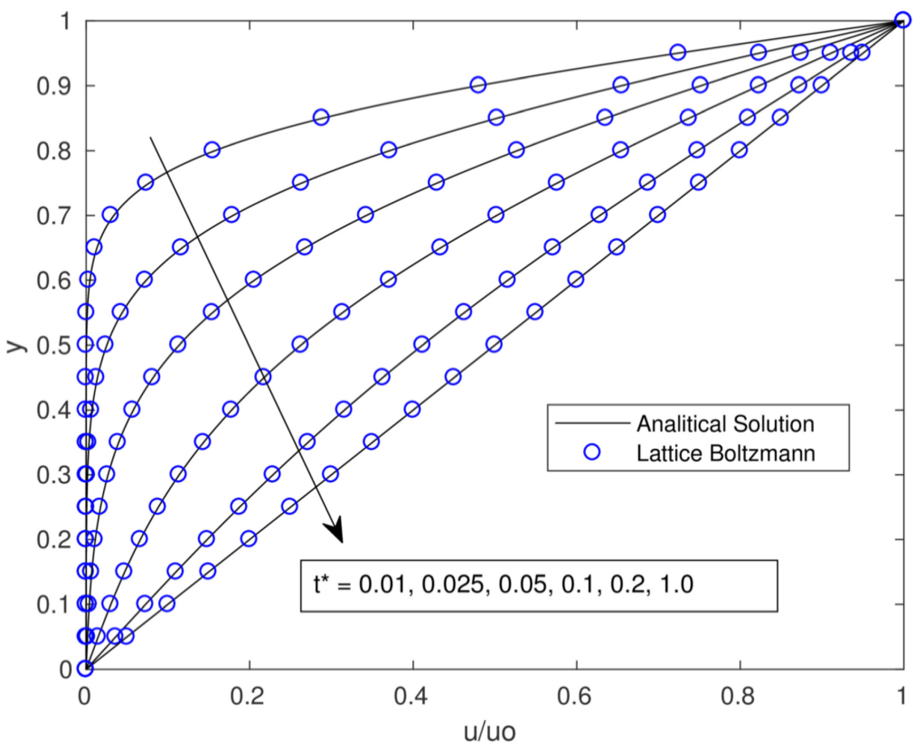
Kinematic viscosity and LBM collision time are connected by

$$v = \left(\tau - \frac{1}{2}\right)\Delta t c_s^2$$

Macroscopic quantities obtained from moments as

$$\rho = \sum f_{\boldsymbol{\alpha}}, \qquad \boldsymbol{u} = \frac{1}{\rho}\sum \boldsymbol{e_\alpha} \cdot f_\alpha, \qquad p = \rho c_s^2$$

Couette Flow – standard LBM



Mach = 0.017

	Dimensionless time t*			
Mesh	0.01	0.025	0.05	0.1
10	-4.06	-4.74	-5.29	-6.65
20	-5.44	-6.12	-6.68	-8.01
40	-6.82	-7.51	-8.07	-9.39
Order of accuracy ~	2.0	2.0	2.0	2.0

Mach = 0.17

	Dimensionless time t*			
Mesh	0.01	0.025	0.05	0.1
10	-4.03	-4.71	-5.26	-6.64
20	-5.37	-6.06	-6.61	-7.97
40	-6.7	-7.39	-7.39	-9.22
Order of accuracy ~	1.9	1.9	1.9	1.9

Reynolds number Re=100

LBM in generalized curvilinear coordinates

The LB equation can be transformed into a generalized curvilinear coordinate system in which the physical and computational planes are represented by (x, y) and (ξ, η), respectively,

$$\xi = \xi(x, y),$$

$$\eta = \eta(x, y).$$

To transform the LBE from a physical plane (x, y) to a computational plane (ξ, η) we must apply

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial x},$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial x}.$$

LBM in generalized curvilinear coordinates

The convection term can be rewritten as

$$\boldsymbol{e}_{\boldsymbol{\alpha}} \cdot \nabla f_{\boldsymbol{\alpha}} = e_{\alpha x}\frac{\partial f_\alpha}{\partial x} + e_{\alpha y}\frac{\partial f_\alpha}{\partial y},$$

$$= e_{\alpha x}\left(\frac{\partial f_\alpha}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial f_\alpha}{\partial \eta}\frac{\partial \eta}{\partial x}\right) + e_{\alpha y}\left(\frac{\partial f_\alpha}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial f_\alpha}{\partial \eta}\frac{\partial \eta}{\partial y}\right),$$

$$= \left(e_{\alpha x}\frac{\partial \xi}{\partial x} + e_{\alpha y}\frac{\partial \xi}{\partial y}\right)\frac{\partial f_\alpha}{\partial \xi} + \left(e_{\alpha x}\frac{\partial \eta}{\partial x} + e_{\alpha y}\frac{\partial \eta}{\partial y}\right)\frac{\partial f_\alpha}{\partial \eta},$$

$$= \tilde{e}_{\alpha\xi}\frac{\partial f_\alpha}{\partial \xi} + \tilde{e}_{\alpha\eta}\frac{\partial f_\alpha}{\partial \eta}.$$

LBM in generalized curvilinear coordinates

Hence, the LB equation can be expressed in the computational domain as

$$\frac{\partial f_\alpha}{\partial t} + \tilde{e}_{\alpha\xi}\frac{\partial f_\alpha}{\partial \xi} + \tilde{e}_{\alpha\eta}\frac{\partial f_\alpha}{\partial \eta} = -\frac{1}{\tau}\left(f_\alpha - f_\alpha^{eq}\right).$$

The relationship between the physical and computational domain satisfies

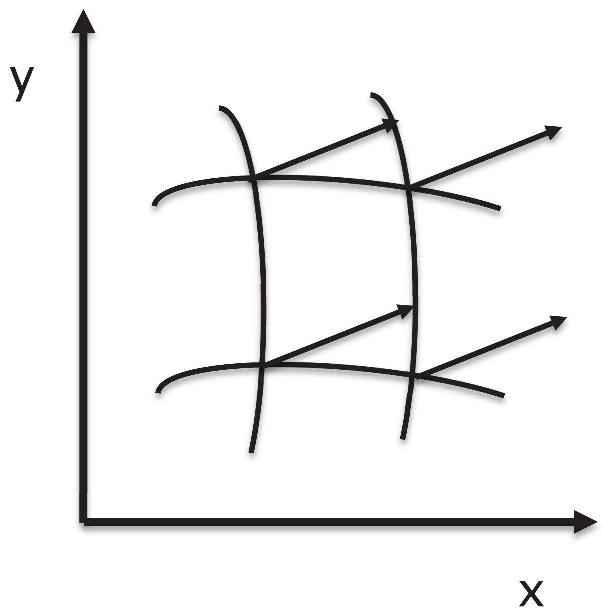
$$\begin{bmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{bmatrix} = \frac{1}{J}\begin{bmatrix} y_\eta & -x_\eta \\ -y_\xi & x_\xi \end{bmatrix}.$$

The Jacobian, J, of the transformation is

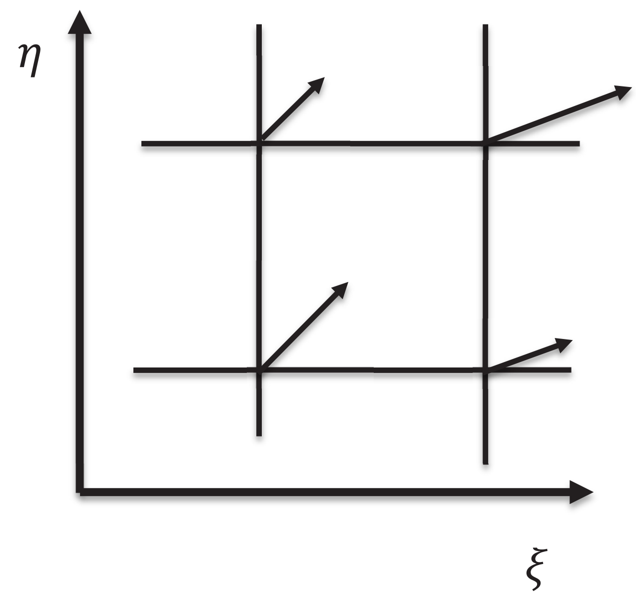
$$J = x_\xi y_\eta - x_\eta y_\xi.$$

LBM in generalized curvilinear coordinates

Physical Domain



Computational Domain



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Curvilinear method construction

A 2nd-order-accurate central discretization for the spatial derivatives is used to approximate

$$R_{\alpha_{i,j}} = -\left(\tilde{e}_{\alpha\xi_{(i,j)}} \frac{f_{\alpha_{(i+1,j)}} - f_{\alpha_{(i-1,j)}}}{2\Delta\xi} + \tilde{e}_{\alpha\eta_{(i,j)}} \frac{f_{\alpha_{(i,j+1)}} - f_{\alpha_{(i,j-1)}}}{2\Delta\eta}\right) - \frac{1}{\tau}\left(f_{\alpha_{(i,j)}} - f^{eq}_{\alpha_{(i,j)}}\right).$$

The solution is advanced in the time by using an explicit four-stage Runge-Kutta scheme:

$$\begin{aligned} f_\alpha^1 &= f_\alpha^t, \\ f_\alpha^2 &= f_\alpha^1 + \frac{\Delta t}{4} R_\alpha^1, \\ f_\alpha^3 &= f_\alpha^1 + \frac{\Delta t}{3} R_\alpha^2, \\ f_\alpha^4 &= f_\alpha^1 + \frac{\Delta t}{2} R_\alpha^3, \\ f_\alpha^{t+\Delta t} &= f_\alpha^1 + \Delta t R_\alpha^4. \end{aligned}$$

Curvilinear LBM construction

Central schemes can be unstable when non-linearities are present. Therefore a 4th-order artificial dissipation term

$$D = -\epsilon \left((\Delta\xi)^4 \left(\frac{\partial f_\alpha}{\partial \xi^4} \right) + (\Delta\eta)^4 \left(\frac{\partial f_\alpha}{\partial \eta^4} \right) \right)$$

is added in order to stabilize the solution (Hejranfar, 2017).

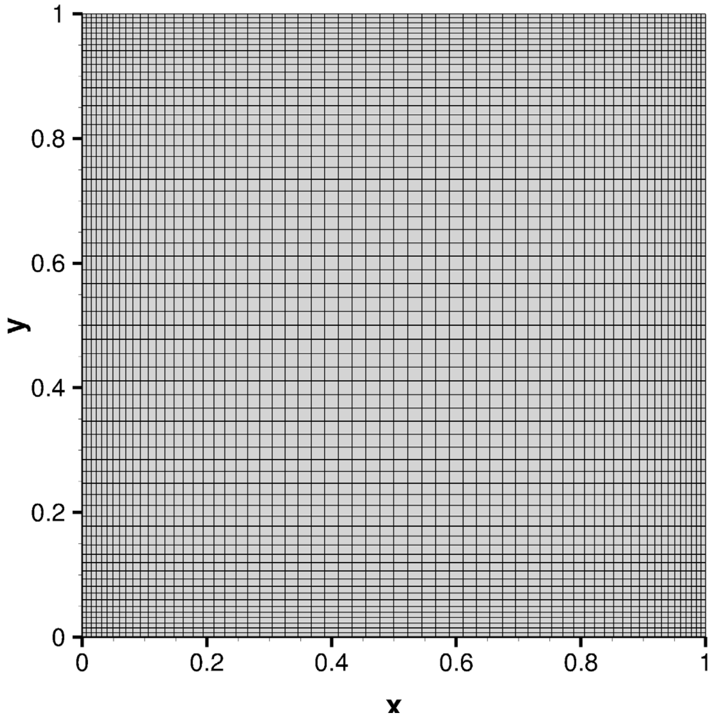
The formal order of the overall scheme remains 2.

K. Hejranfar, M. Hajihassanpour, Chebyshev collocation spectral lattice Boltzmann method in generalized curvilinear coordinates, Computers and Fluids 146 (2017) 154–173.

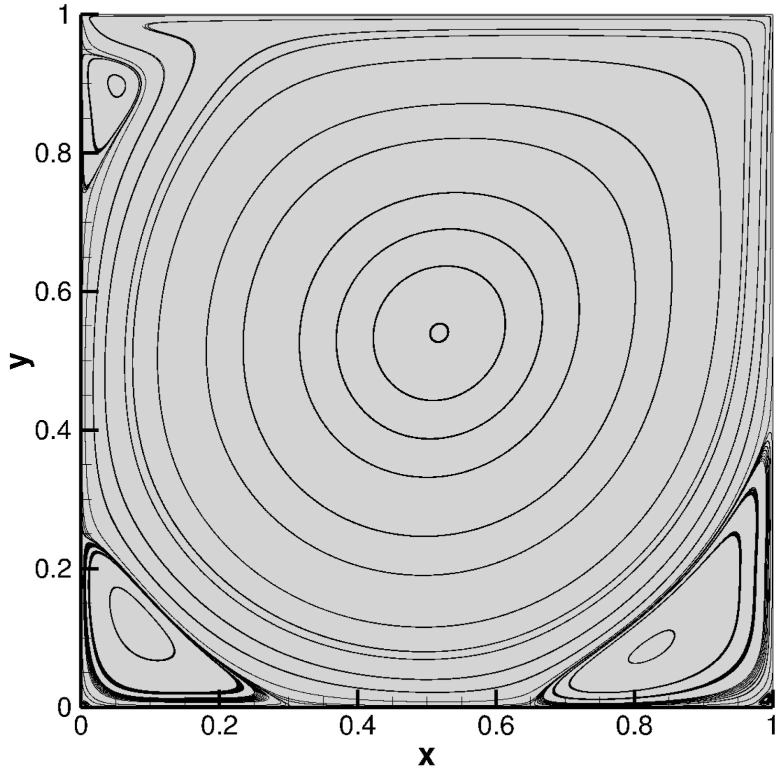
Verification – Lid-driven Cavity with stretched mesh

$$x = H\frac{(2\alpha+\beta)[(\beta+1)/(\beta-1)]^{(\xi-\alpha)/(1-\alpha)} + 2\alpha - \beta}{(2\alpha+1)\left(1+[(\beta+1)/(\beta-1)]^{(\xi-\alpha)/(1-\alpha)}\right)}$$

$$y = H\frac{(2\alpha+\beta)[(\beta+1)/(\beta-1)]^{(\eta-\alpha)/(1-\alpha)} + 2\alpha - \beta}{(2\alpha+1)\left(1+[(\beta+1)/(\beta-1)]^{(\eta-\alpha)/(1-\alpha)}\right)}$$



Re = 3200



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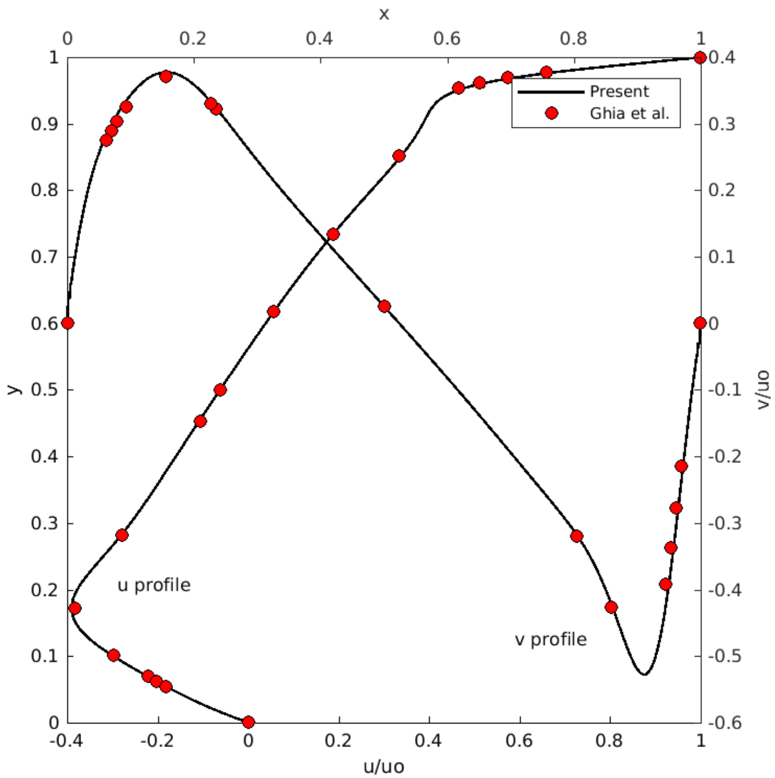
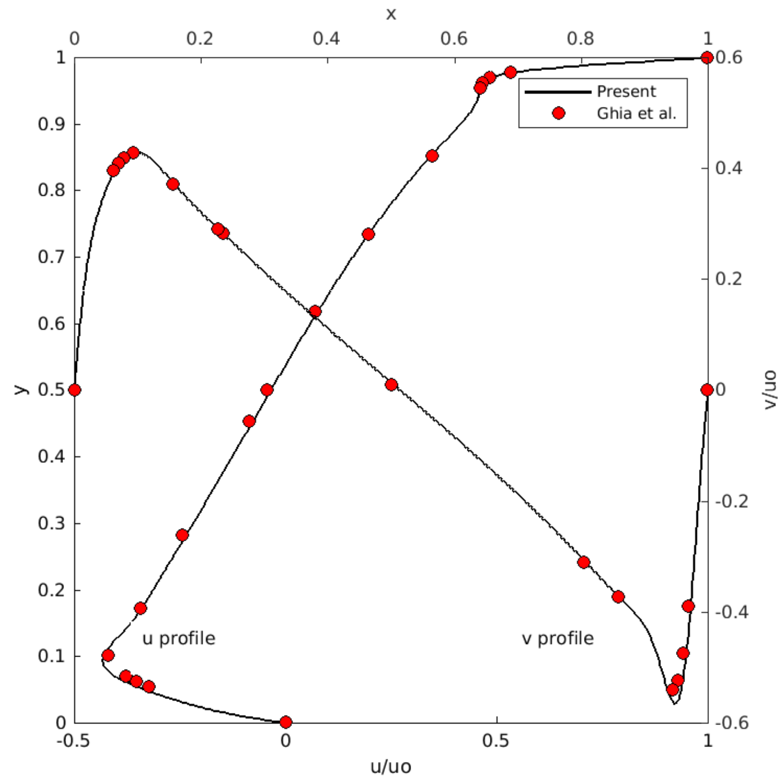
2D Lid-driven cavity

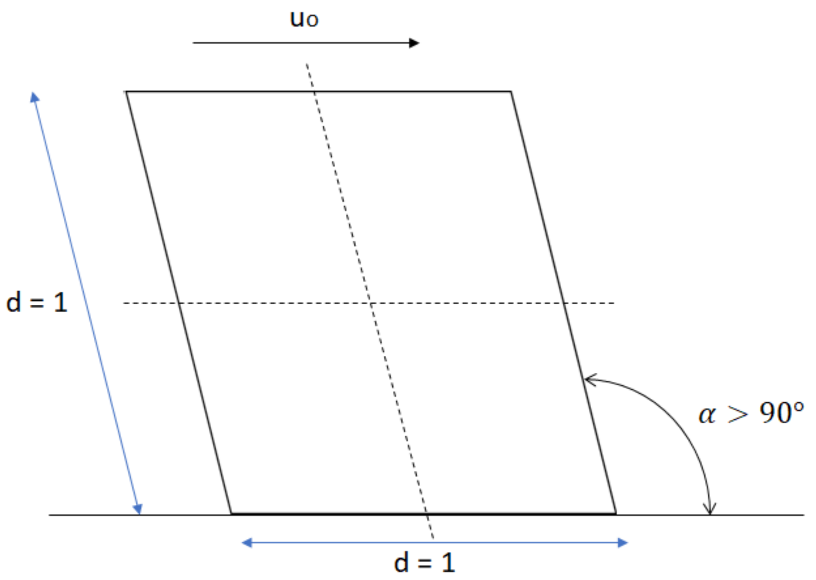
Re = 1000

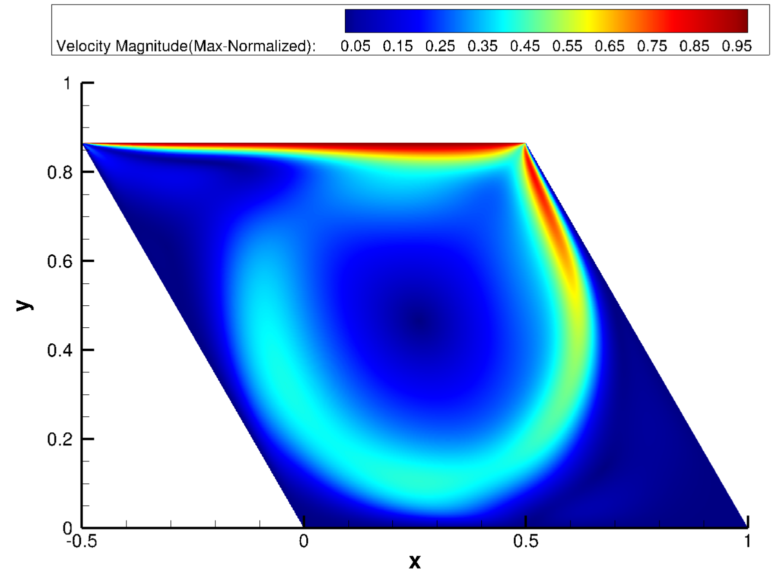


Re = 3200



Skewed cavity





$$x(\xi, \eta) = \xi + \eta \cos \alpha \qquad x_\xi = 1, \quad y_\xi = 0$$

$$y(\xi, \eta) = \eta \sin \alpha \qquad x_\eta = \cos \alpha, \quad y_\eta = sin\alpha$$

$$J = \sin \alpha$$

Re = 1000, α = 120

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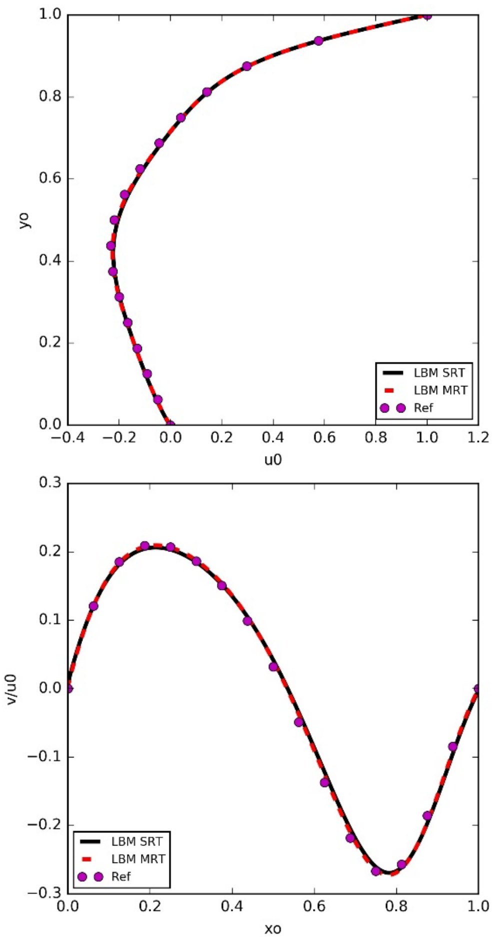
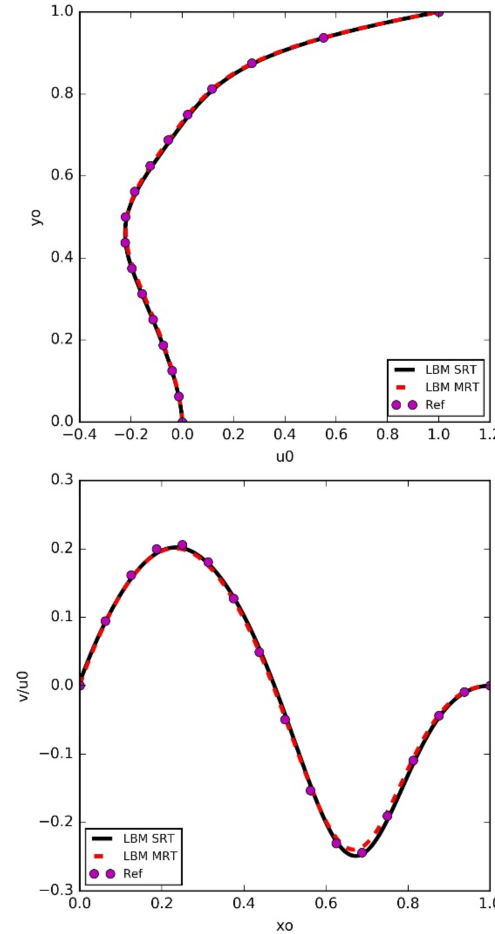
Skewed cavity – Re=100

$\alpha = 120$

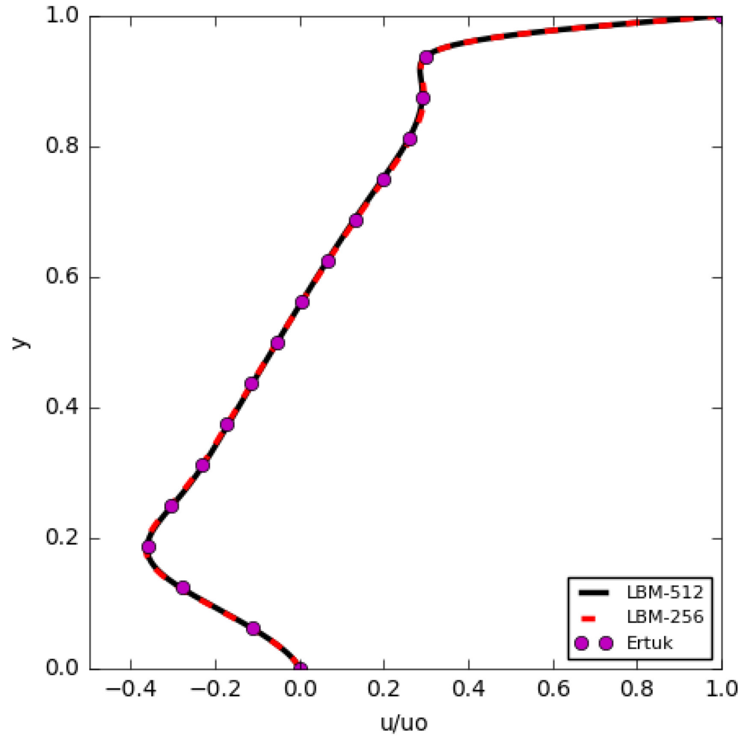


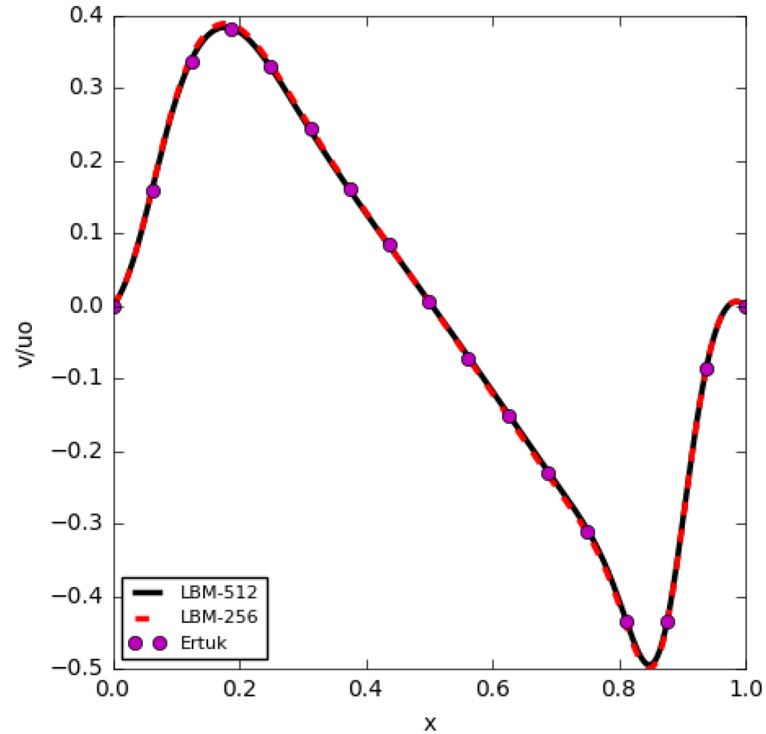
$\alpha = 135$



Verification for single- as well as multi-time relaxation collision operator

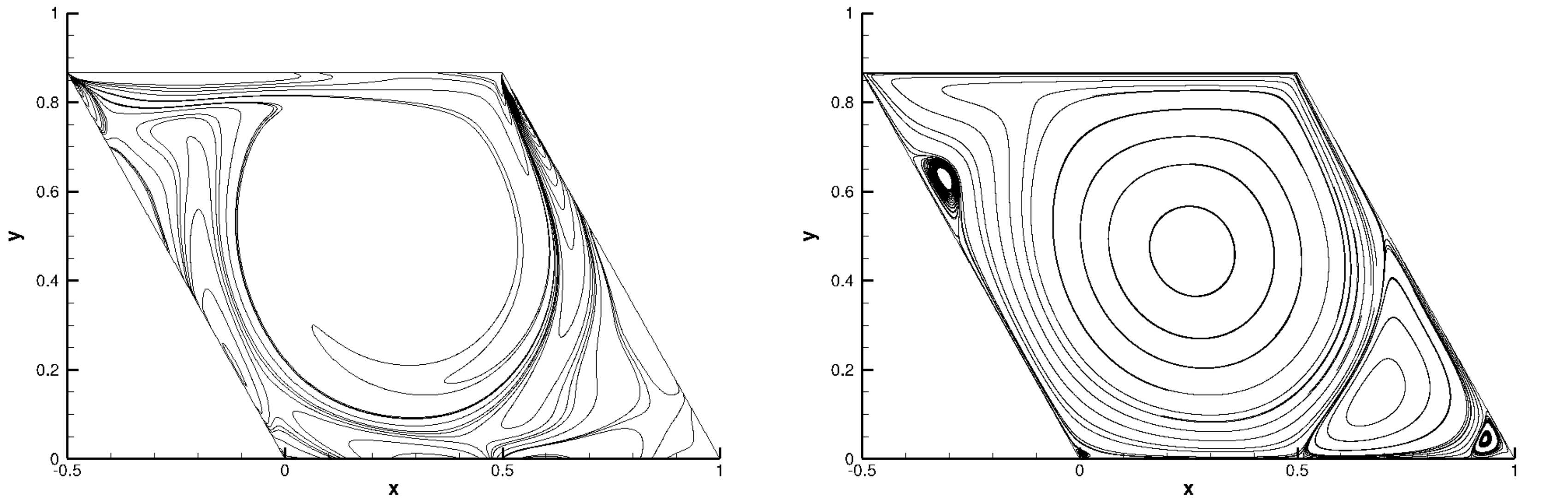
Skewed Cavity – Re=1000, $\alpha = 120$





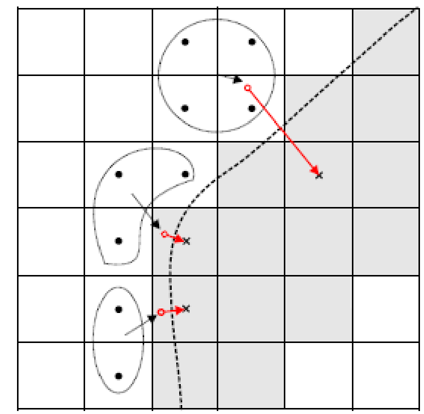
Grid sensitivity

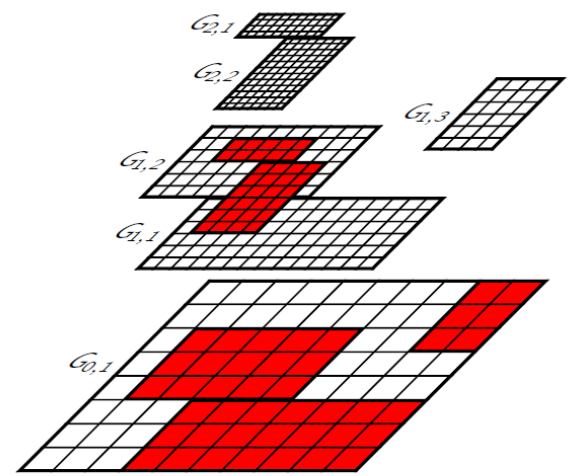
Skewed Cavity – Re=1000, α = 120



AMROC-LBM software

- Finite volume based LBM implementation
- Boundary representation with signed distance level set function
- Bouzidi-type fixed wall boundary conditions
- One-sided interpolation/extrapolation of macroscopic quantities at embedded boundaries
- Block-structured adaptive mesh refinement (Deiterding, 2011)
- Mesh adaptation algorithm for LBM is mathematically equivalent to method by Chen et al. (as in Powerflow)
- Verified and validated (Deiterding and Wood, 2016)
- Also very large number of finite volume patch solvers -> www.vtf.website

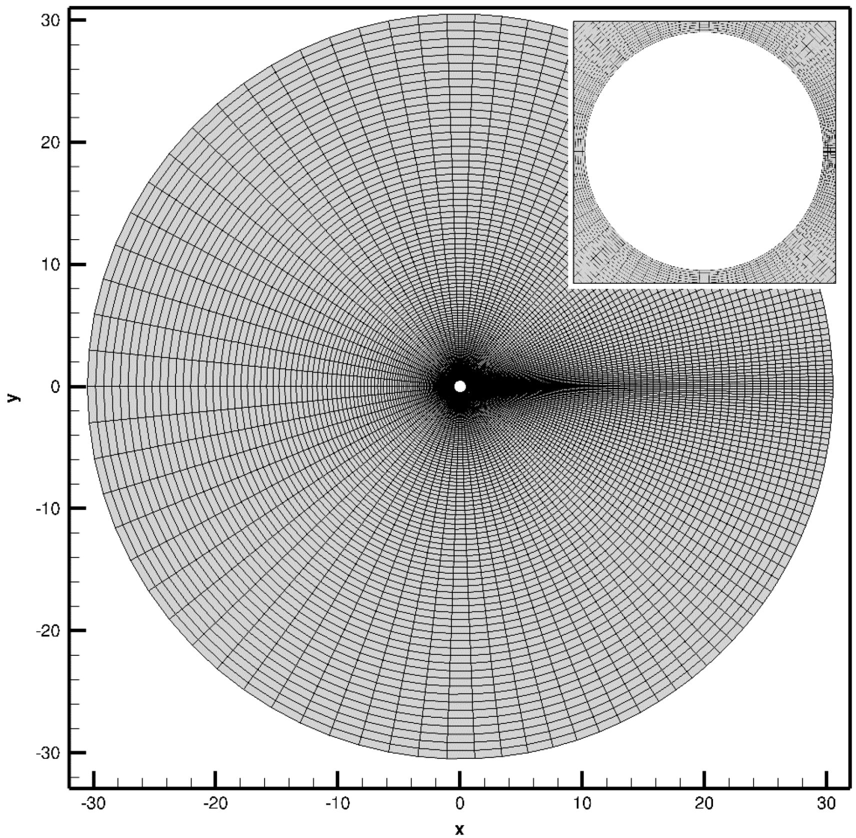




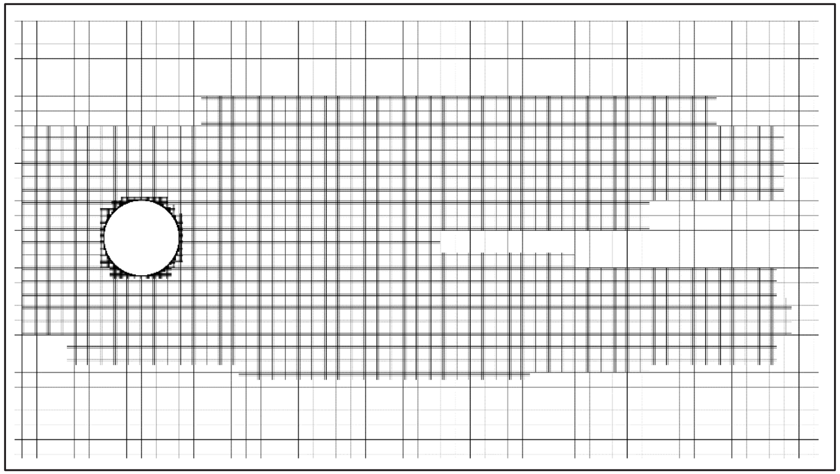
R. Deiterding. Block-structured adaptive mesh refinement - theory, implementation and application. European Series in Applied and Industrial Mathematics: Proceedings, 34:97-150 (2011).

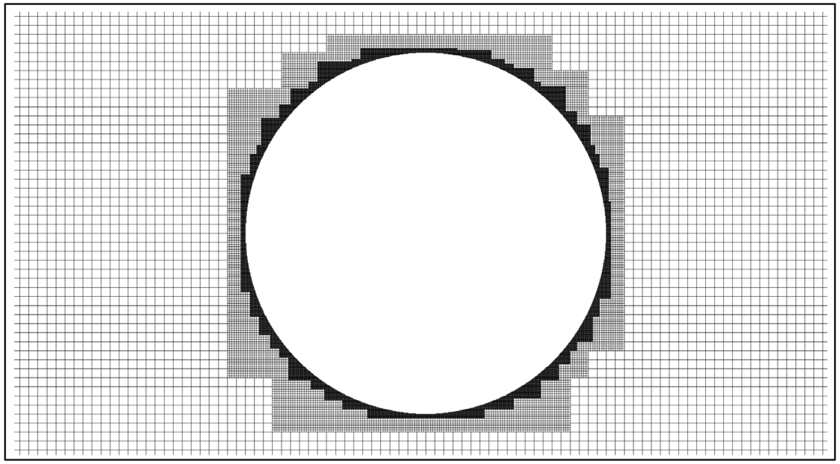
R. Deiterding and S. L. Wood. In Results in Numerical and Experimental Fluid Mechanics X, volume 132 of Notes on Numerical Fluid Mechanics and Multidisciplinary Design, Springer, pages 845-857 (2016).

2D circular cylinder



Mesh for present method





Snapshot of AMROC-LBM mesh

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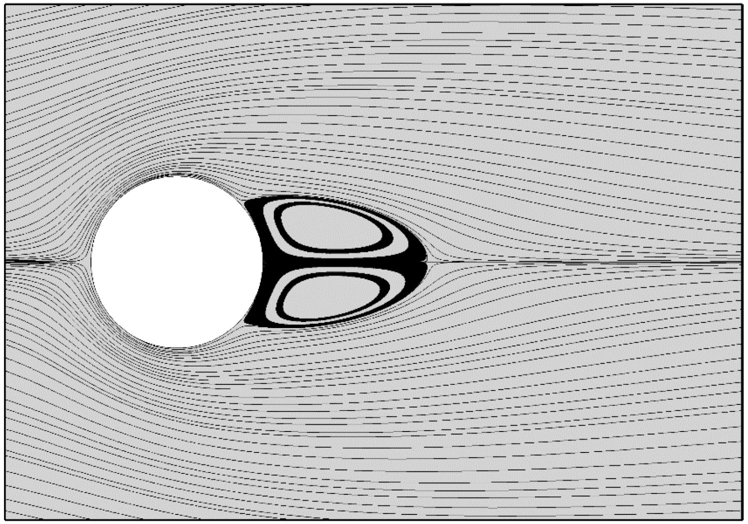
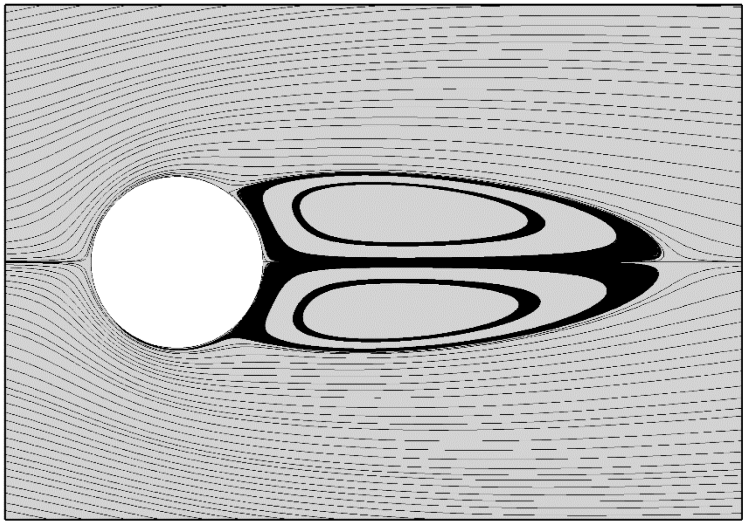
Steady state flow cases

Re = 20



Re = 40



The predicted length of the wake, *L*, is an important output

Normalized wake length: 2L/D

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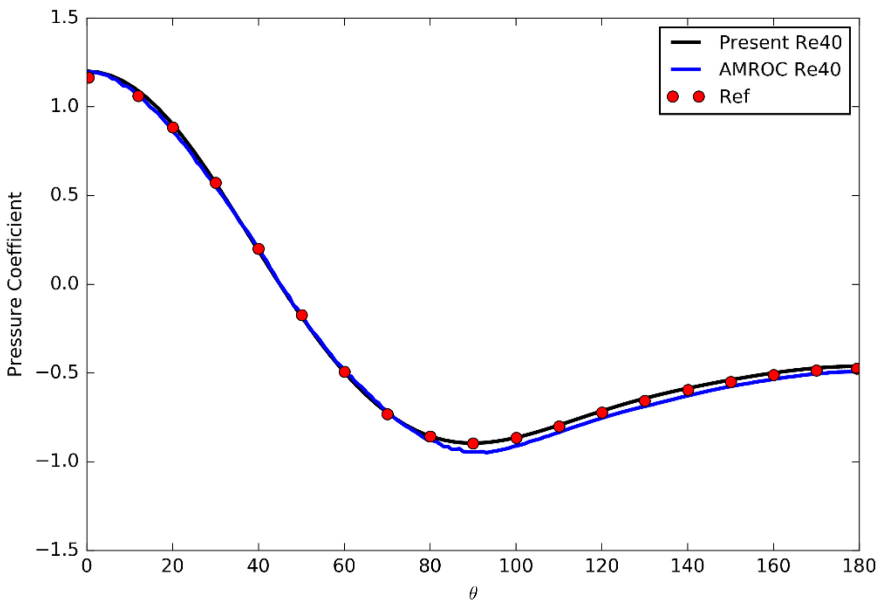
Steady state flow cases

Pressure coefficient distribution around body



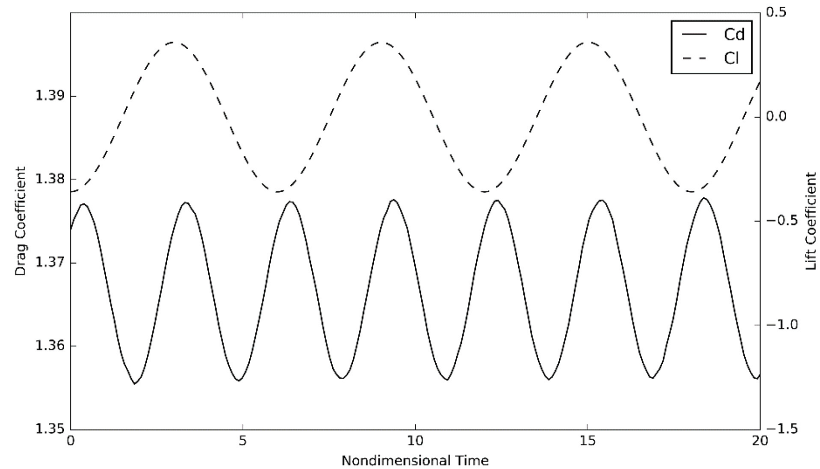
Effect of Re on steady flow over the circular cylinder

Re	Author(s)	Cd	Cp(0)	Cp(180)	2L/D
20	Tritton (1959), Exp.	2.10	-	-	0.6
	Henderson (1995)	2.06	-	-	-
	Dennis and Chang (1970)	2.05	1.27	-0.58	1.88
	Hejranfar & Ezzatneshan (2014)	2.02	1.25	-0.59	1.84
	He et al.	2.15	1.28	-0.58	1.84
	AMROC-LBM	1.98	1.26	-0.59	1.85
	Present	2.02	1.31	-0.55	1.85
40	Tritton (1959) , Exp.	1.59	-	-	-
	Henderson (1995)	1.55	-	-0.53	-
	Dennis and Chang (1970)	1.52	1.14	-0.5	4.69
	Hejranfar & Ezzatneshan (2014)	1.51	1.15	-0.48	4.51
	He et al.	1.49	1.11	-0.48	4.49
	AMROC-LBM	1.45	1.19	-0.49	4.66
	Present	1.51	1.18	-0.5	4.32

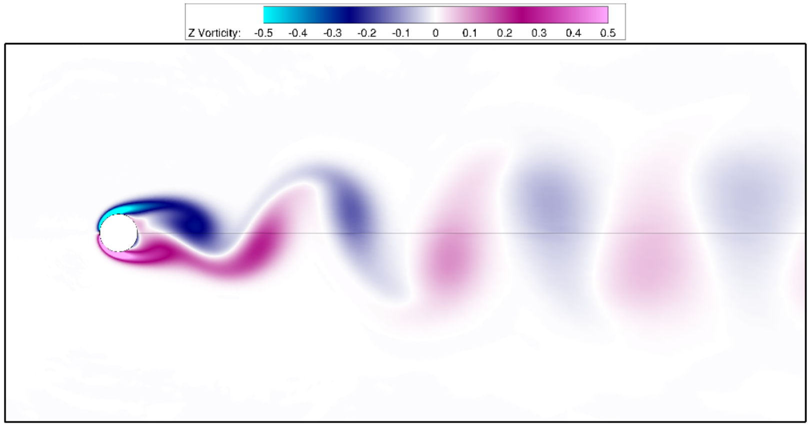
Present method in excellent agreement with references

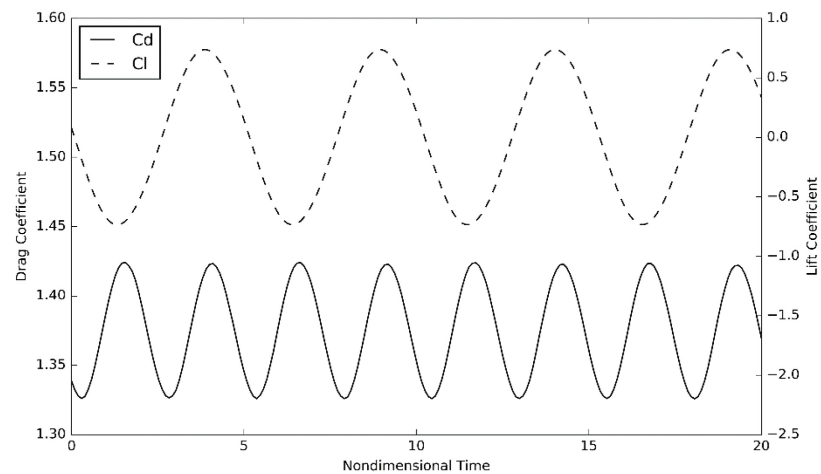
Unsteady flow cases

Re	Author(s)	St	$\overline{Cd}$	Cl'
100	Chiu et al. (2010)	0.167	1.35	0.30
	AMROC-LBM	0.166	1.28	0.32
	Present	0.165	1.36	0.35
200	Chiu et al. (2010)	0.198	1.37	0.71
	AMROC-LBM	0.196	1.26	0.70
	Present	0.196	1.37	0.73



Re = 100

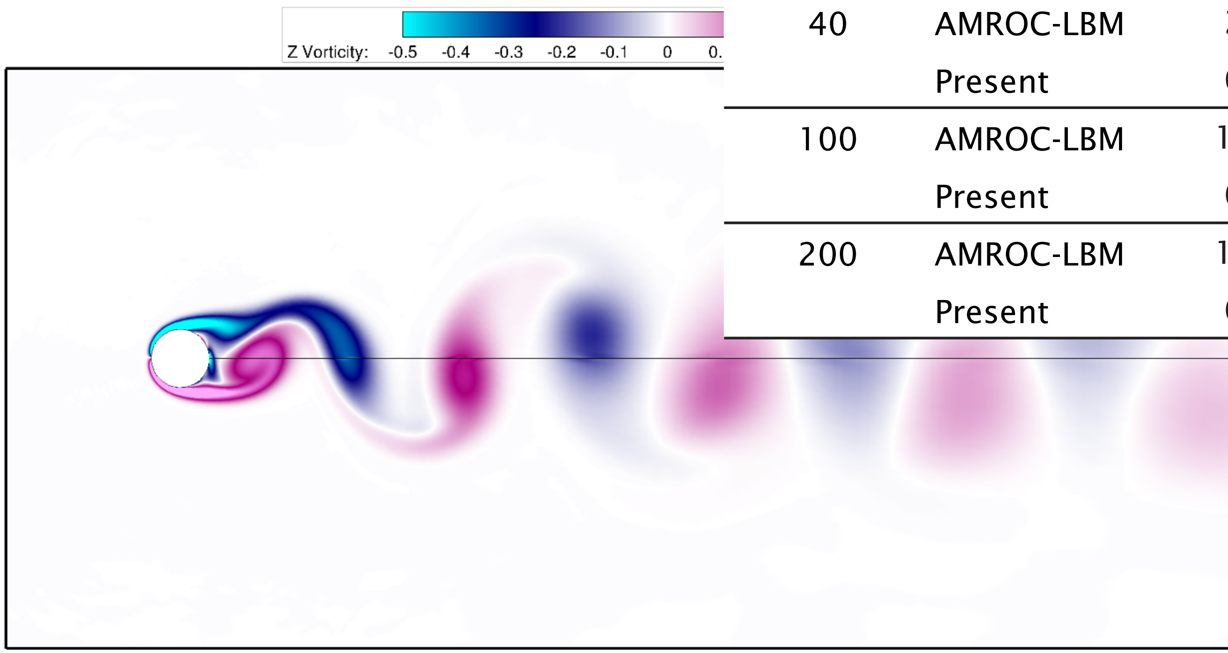




P. H. Chiu, R. K. Lin, T. W. Sheu, A differentially interpolated direct forcing immersed boundary method for predicting incompressible Navier Stokes equations in time-varying complex geometries, Journal of Computational Physics 229 (2010) 4476–4500.

Unsteady flow cases – computational performance

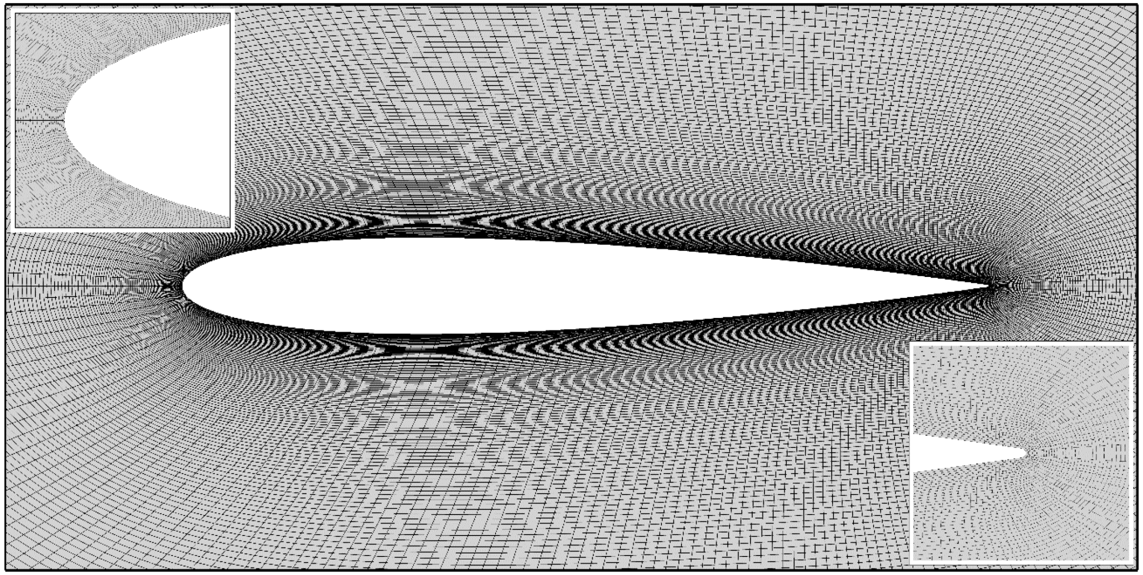
Re = 200



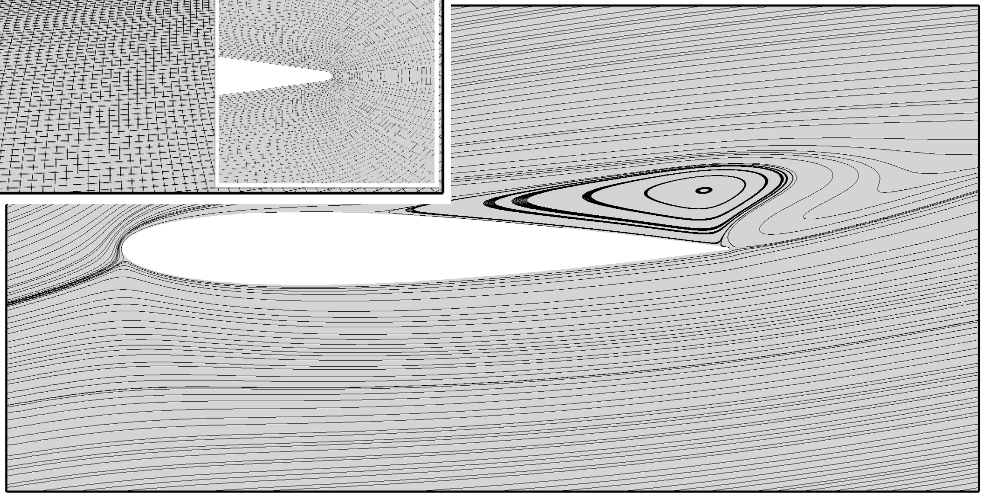
Re		CPU time	Mesh size
20	AMROC-LBM	24:55:21	297,796
	Present	06:08:41	65,536
40	AMROC-LBM	27:10:08	317,732
	Present	05:57:17	65,536
100	AMROC-LBM	113:15:37	1,026,116
	Present	05:58:49	65,536
200	AMROC-LBM	130:37:18	1,130,212
	Present	06:03:42	65,536

Computations were run on University of Southampton IRIDIS 4 cluster

2D NACA0012 airfoil

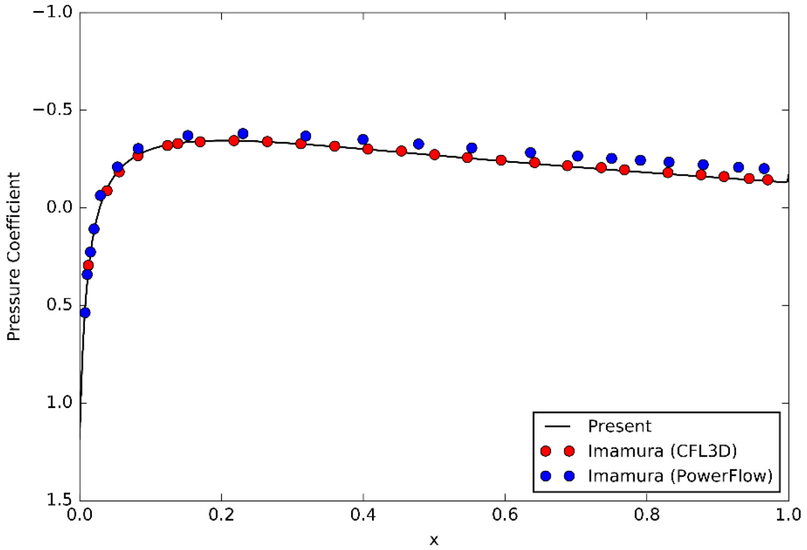


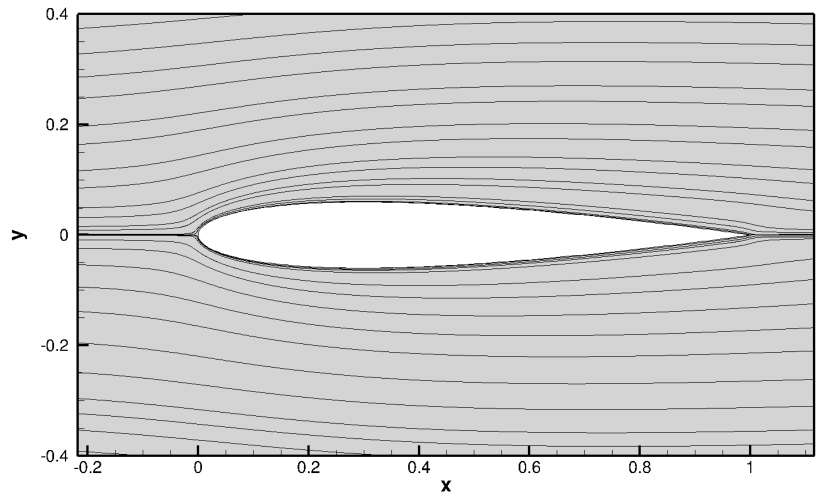
- Streamlines
- Re = 500
- Angle of attack 10º
- Note flawless flow at trailing edge!



O-grid with domain radius of 15 chord lengths

2D NACA0012 airfoil – AOA=0

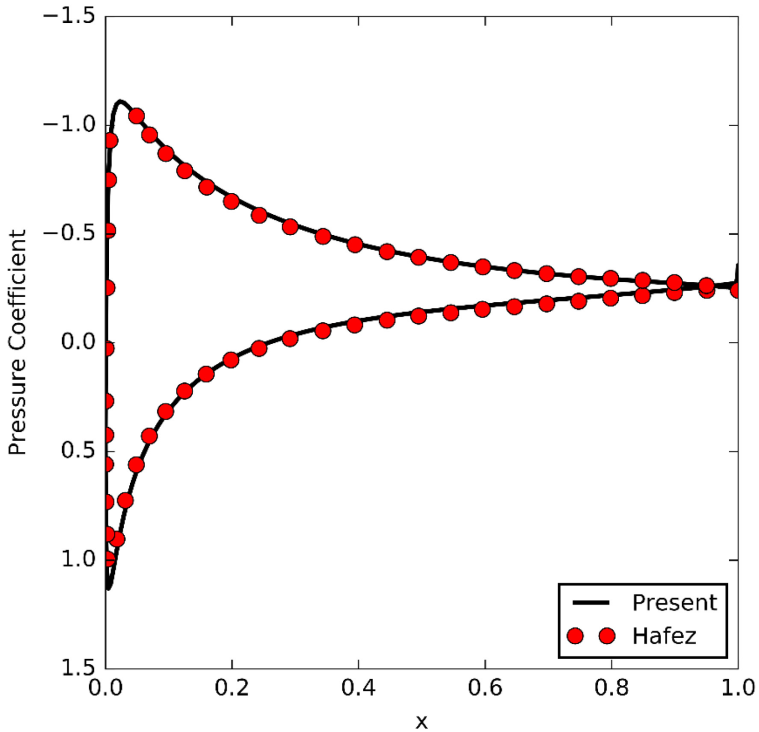


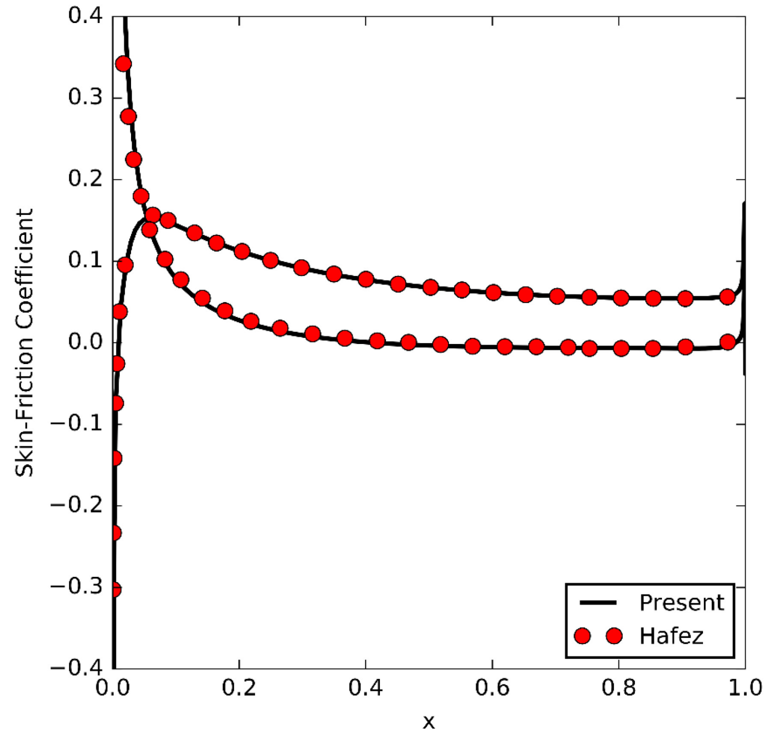


T. Imamura, K. Suzuki. Flow Simulation Around an Airfoil by Lattice Boltzmann Method on Generalized Coordinates. AIAA, 43 (2005) 1–6.

Verification for angle of attack = 0, Re = 500

2D NACA0012 airfoil, AOA=10





M. Hafez, A. Shatalov, M. Nakajima, Improved numerical simulations of incompressible flows based on viscous/inviscid interaction procedures, Computers & Fluids 36 (2007) 1588–1591.

Verification for AOA = 10, Re = 500

Conclusions and outlook

- A 2^{nd}-order accurate 2D lattice Boltzmann BGK method on structured non-Cartesian meshes has been formulated and verified
- Approach is built on mapping of the streaming step
- Discretization of the moment space, i.e., the collision operator is unchanged compared to the standard LBM-BGK scheme
- Comparisons to our Cartesian, but non-uniform, solver system AMROC-LBM are very favourable
- Evaluation of quantities on bodies in flow is unambiguous and very accurate
- Current method is time-explicit and will lead to very small global time steps from near-body cells -> consider time-implicit integration in wall-normal direction
- Hybrid chimera-type meshing currently under investigation to eventually combine both approaches in the future

References

- Henderson, R. D. Details of the drag curve near the onset of vortex shedding. Phys. Fluids, (1995).
- F. Nieuwstadt, H.B. Keller. Viscous flow past circular cylinders, Comput. Fluids 59 (1973).
- G. Eitel-Amor, M. Meinke, W. Schroeder. A lattice-Boltzmann method with hierarchically refined meshes, (2013).
- Zhang H-Q, Fey U, Noack BR, Koenig M, Eckelmann H. On the transition of the cylinder wake, (1995).
- U. Ghia, K. Ghia, C. Shin, High-Re solutions for incompressible flow using the naiver-stokes equations and a multigrid method, Journal of Computational Physics (1982).
- Ercan Erturk, B. Dursun, Numerical solutions of 2-D steady incompressible flow in a driven skewed cavity, Journal of Applied Mathematics and Mechanics, (2007).