Lecture 8

Supplementary material: Using the SAMR approach for elliptic problems

Course *Block-structured Adaptive Finite Volume Methods for Shock-Induced Combustion Simulation*

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Outline

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Poisson equation

$$\begin{aligned} \Delta q(\mathbf{x}) &= \psi(\mathbf{x})\,, \quad \mathbf{x} \in \Omega \subset \mathbb{R}^d, \quad q \in \mathrm{C}^2(\Omega), \quad \psi \in \mathrm{C}^0(\Omega) \\ q &= \psi^\Gamma(\mathbf{x})\,, \quad \mathbf{x} \in \partial\Omega \end{aligned}$$

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Discrete Poisson equation in 2D:

$$\frac{Q_{j+1,k} - 2Q_{jk} + Q_{j-1,k}}{\Delta x_1^2} + \frac{Q_{j,k+1} - 2Q_{jk} + Q_{j,k-1}}{\Delta x_2^2} = \psi_{jk}$$

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Operator

$$\mathcal{A}(Q_{\Delta x_1,\Delta x_2}) = \begin{bmatrix} & \frac{1}{\Delta x_2^2} & \\ \frac{1}{\Delta x_1^2} & -\left(\frac{2}{\Delta x_1^2} + \frac{2}{\Delta x_2^2}\right) & \frac{1}{\Delta x_2^2} \\ & \frac{1}{\Delta x_2^2} & \end{bmatrix} Q(x_{1,j}, x_{2,k}) = \psi_{jk}$$

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$$Q_{jk} = \frac{1}{\sigma}\left[(Q_{j+1,k} + Q_{j-1,k})\Delta x_2^2 + (Q_{j,k+1} + Q_{j,k-1})\Delta x_1^2 - \Delta x_1^2 \Delta x_2^2 \psi_{jk}\right]$$

with $\sigma = \dfrac{2\Delta x_1^2 + 2\Delta x_2^2}{\Delta x_1^2 \Delta x_2^2}$

Iterative methods

Jacobi iteration

$$Q_{jk}^{m+1} = \frac{1}{\sigma}\left[(Q_{j+1,k}^{m} + Q_{j-1,k}^{m})\Delta x_2^2 + (Q_{j,k+1}^{m} + Q_{j,k-1}^{m})\Delta x_1^2 - \Delta x_1^2 \Delta x_2^2 \psi_{jk}\right]$$

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Efficient parallelization / patch-wise application not possible!

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Checker-board or Red-Black Gauss Seidel iteration

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   for $j + k \mod 2 = 0$
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   for $j + k \mod 2 = 1$

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Gauss-Seidel methods require $\sim 1/2$ of iterations than Jacobi method, however, iteration count still proportional to number of unknowns [Hackbusch, 1994]

Smoothing vs. solving

ν iterations with iterative linear solver

$$Q^{m+\nu} = \mathcal{S}(Q^m, \psi, \nu)$$

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Defect after m iterations

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Defect after $m + \nu$ iterations

$$d^{m+\nu} = \psi - \mathcal{A}(Q^{m+\nu}) = \psi - \mathcal{A}(Q^m + v_\nu^m) = d^m - \mathcal{A}(v_\nu^m)$$

with correction

$$v_\nu^m = \mathcal{S}(\vec{0}, d^m, \nu)$$

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Neglecting the sub-iterations in the smoother we write

$$Q^{n+1} = Q^n + v = Q^n + \mathcal{S}(d^n)$$

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Observation: Oscillations are damped faster on coarser grid.

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Coarse grid correction:

$$Q^{n+1} = Q^n + v = Q^n + \mathcal{P}\mathcal{S}\mathcal{R}(d^n)$$

where $\mathcal{R}$ is suitable restriction operator and $\mathcal{P}$ a suitable prolongation operator

Two-grid correction method

Relaxation on current grid:

$$\bar{Q} = \mathcal{S}(Q^n, \psi, \nu)$$

$$Q^{n+1} = \bar{Q} + \mathcal{P}\mathcal{S}(\vec{0}, \cdot, \mu)\mathcal{R}(\psi - \mathcal{A}(\bar{Q}))$$

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Algorithm:

$$\begin{aligned} &\bar{Q} := \mathcal{S}(Q^n, \psi, \nu) \\ &d := \psi - \mathcal{A}(\bar{Q}) \end{aligned}$$

$$\begin{aligned} &d_c := \mathcal{R}(d) \\ &v_c := \mathcal{S}(0, d_c, \mu) \\ &v := \mathcal{P}(v_c) \\ &Q^{n+1} := \bar{Q} + v \end{aligned}$$

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Algorithm:	with smoothing:
$\bar{Q} := \mathcal{S}(Q^n, \psi, \nu)$	$d := \psi - \mathcal{A}(Q)$
$d := \psi - \mathcal{A}(\bar{Q})$	$v := \mathcal{S}(0, d, \nu)$
	$r := d - \mathcal{A}(v)$
$d_c := \mathcal{R}(d)$	$d_c := \mathcal{R}(r)$
$v_c := \mathcal{S}(0, d_c, \mu)$	$v_c := \mathcal{S}(0, d_c, \mu)$
$v := \mathcal{P}(v_c)$	$v := v + \mathcal{P}(v_c)$
$Q^{n+1} := \bar{Q} + v$	$Q^{n+1} := Q + v$

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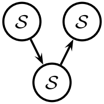
with pre- and post-iteration:

$$\begin{aligned}
d &:= \psi - \mathcal{A}(Q) \\
v &:= \mathcal{S}(0, d, \nu_1) \\
r &:= d - \mathcal{A}(v) \\
d_c &:= \mathcal{R}(r) \\
v_c &:= \mathcal{S}(0, d_c, \mu) \\
v &:= v + \mathcal{P}(v_c) \\
d &:= d - \mathcal{A}(v) \\
r &:= \mathcal{S}(0, d, \nu_2) \\
Q^{n+1} &:= Q + v + r
\end{aligned}$$

[Hackbusch, 1985]

Multi-level methods and cycles

V-cycle
$\gamma = 1$
2-grid



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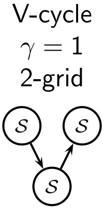

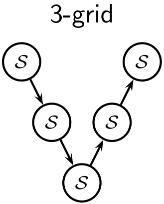

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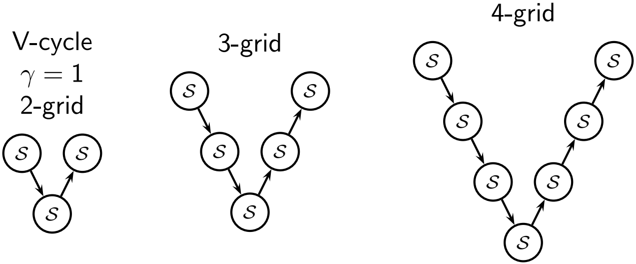
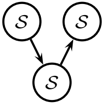


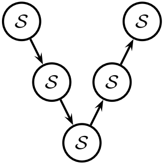
Multi-level methods and cycles

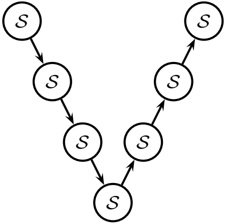
4-grid

V-cycle
$\gamma = 1$
2-grid

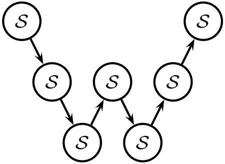


3-grid





W-cycle
$\gamma = 2$



Multi-level methods and cycles

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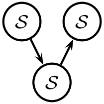

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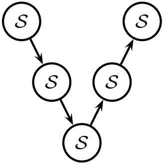

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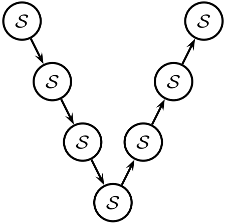

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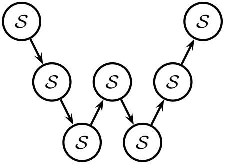

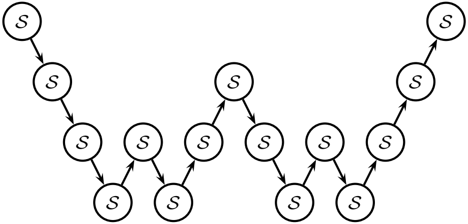

[Hackbusch, 1985] [Wesseling, 1992] ...

Stencil modification at coarse-fine boundaries in 1D

1D Example: Cell j, $\psi - \nabla \cdot \nabla q = 0$

$$d_j^l = \psi_j - \frac{1}{\Delta x_l}\left(\frac{1}{\Delta x_l}(Q_{j+1}^l - Q_j^l) - \frac{1}{\Delta x_l}(Q_j^l - Q_{j-1}^l)\right)$$

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H is approximation to *derivative* of Q^l.

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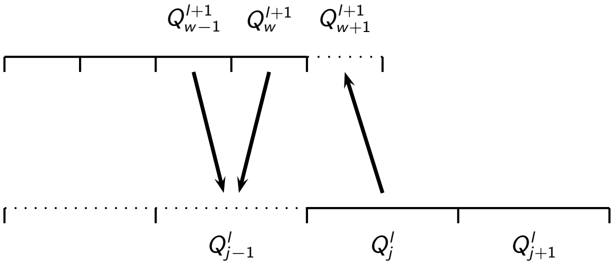
Consider 2-level situation with $r_{l+1} = 2$:



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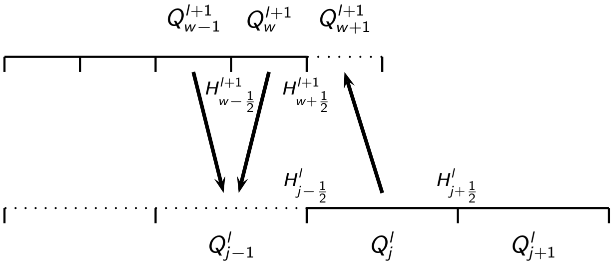

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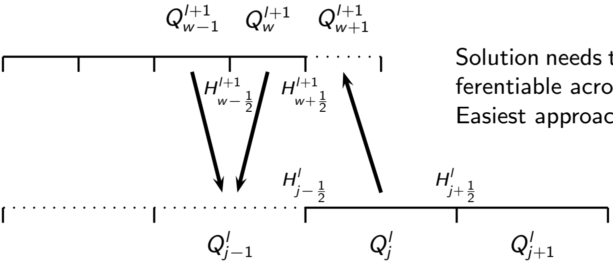

Solution needs to be continuously differentiable across interface.
Easiest approach: $H_{w+\frac{1}{2}}^{l+1} \equiv H_{j-\frac{1}{2}}^l$

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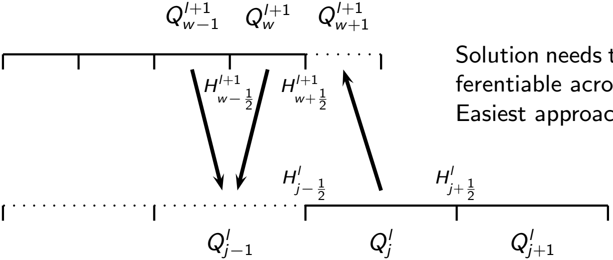

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No specific modification necessary for 1D vertex-based stencils, cf. [Bastian, 1996]

Stencil modification at coarse-fine boundaries in 1D II

Set $H^{l+1}_{w+\frac{1}{2}} = H_{\mathcal{I}}$.

Stencil modification at coarse-fine boundaries in 1D II

Set $H^{l+1}_{w+\frac{1}{2}} = H_{\mathcal{I}}$. Inserting Q gives

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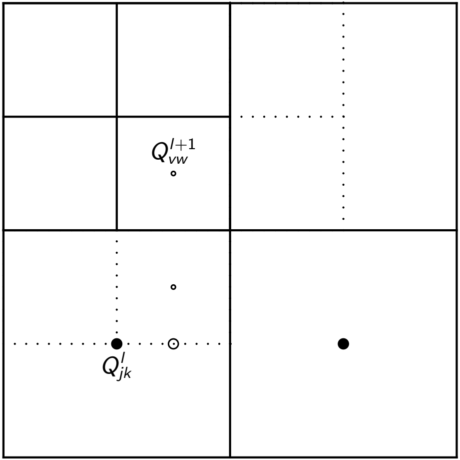
$$\check{d}^{l}_{j} = \psi_j - \frac{1}{\Delta x_l}\left(\frac{1}{\Delta x_l}(Q^{l}_{j+1} - Q^{l}_{j}) - \frac{2}{3\Delta x_{l+1}}(Q^{l}_{j} - Q^{l+1}_{w})\right)$$

Stencil modification at coarse-fine boundaries: 2D



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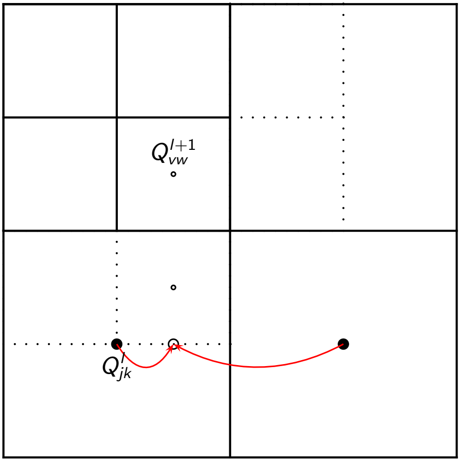

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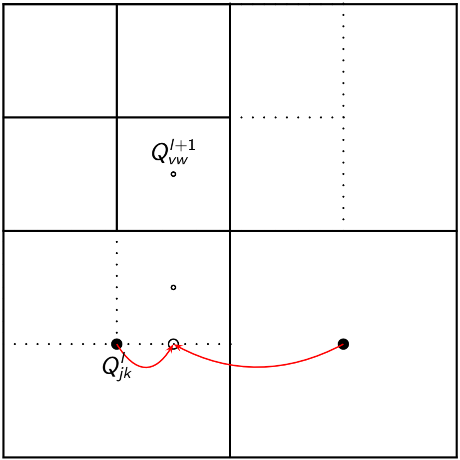

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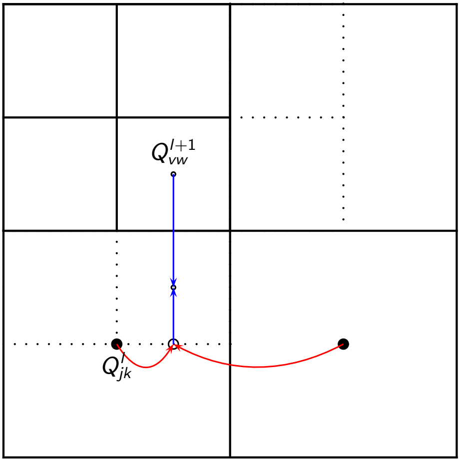

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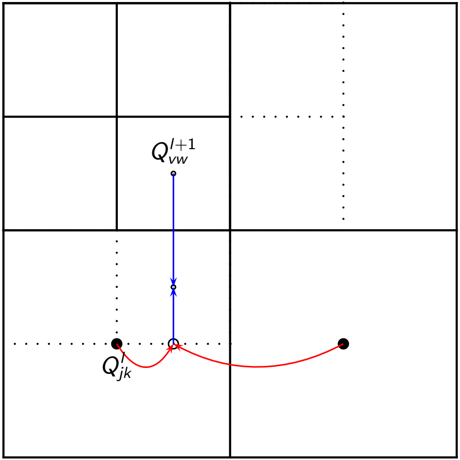

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In general:

$$Q^{l+1}_{v,w-1} = \left(1 - \frac{2}{r_{l+1}+1}\right) Q^{l+1}_{vw} + \frac{2}{r_{l+1}+1}\left((1-f)Q^{l}_{jk} + fQ^{l}_{j+1,k}\right)$$

with

$$f = \frac{x^{v}_{1,l+1} - x^{j}_{1,l}}{\Delta x_{1,l}}$$

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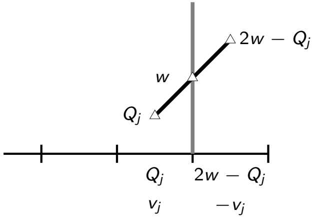

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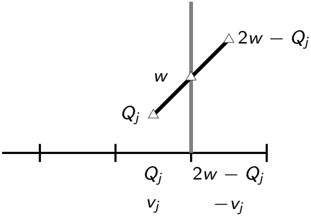

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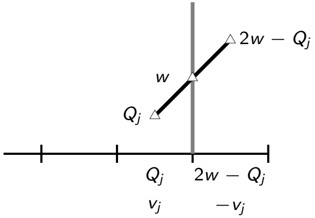

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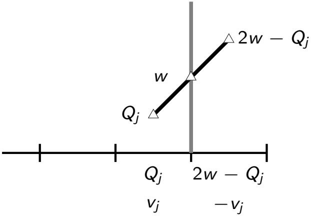

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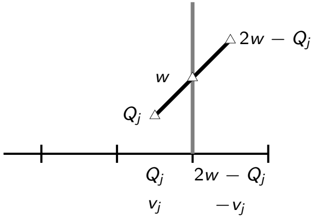

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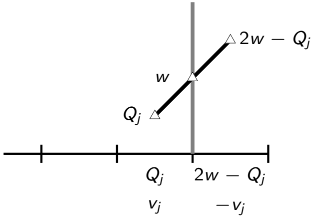

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- Looping instead of time steps and check of convergence

Additive geometric multigrid algorithm

AdvanceLevelMG(l) - Correction Scheme

Set ghost cells of Q^l	
Calculate defect d^l from Q^l, ψ^l	$d^l := \psi^l - \mathcal{A}(Q^l)$
If $(l < l_{max})$	
Calculate updated defect r^{l+1} from v^{l+1}, d^{l+1}	$r^{l+1} := d^{l+1} - \mathcal{A}(v^{l+1})$
Restrict d^{l+1} onto d^l	$d^l := \mathcal{R}_l^{l+1}(r^{l+1})$
Do ν_1 smoothing steps to get correction v^l	$v^l := \mathcal{S}(0, d^l, \nu_1)$
If $(l > l_{min})$	
Do $\gamma > 1$ times	
AdvanceLevelMG($l-1$)	
Set ghost cells of v^{l-1}	
Prolongate and add v^{l-1} to v^l	$v^l := v^l + \mathcal{P}_l^{l-1}(v^{l-1})$
If $(\nu_2 > 0)$	
Set ghost cells of v^l	
Update defect d^l according to v^l	$d^l := d^l - \mathcal{A}(v^l)$
Do ν_2 post-smoothing steps to get r^l	$r^l := \mathcal{S}(v^l, d^l, \nu_2)$
Add addional correction r^l to v^l	$v^l := v^l + r^l$
Add correction v^l to Q^l	$Q^l := Q^l + v^l$

Additive Geometric Multiplicative Multigrid Algorithm

`Start - Start iteration on level` l_{max}

    `For` $l = l_{max}$ `Downto` $l_{min} + 1$ `Do`
        `Restrict` Q^l `onto` Q^{l-1}
    `Regrid(0)`
    `AdvanceLevelMG(`l_{max}`)`

See also: [Trottenberg et al., 2001], [Canu and Ritzdorf, 1994]
Vertex-based: [Brandt, 1977], [Briggs et al., 2001]

Example

On $\Omega = [0, 10] \times [0, 10]$ use hat function

$$\psi = \begin{cases} -A_n \cos\left(\frac{\pi r}{2R_n}\right) \ , & r < R_n \\ 0 & \text{elsewhere} \end{cases}$$

with $r = \sqrt{(x_1 - X_n)^2 + (x_2 - Y_n)^2}$ to define three sources with

n	A_n	R_n	X_n	Y_n
1	0.3	0.3	6.5	8.0
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3	-0.1	0.4	7.0	3.0

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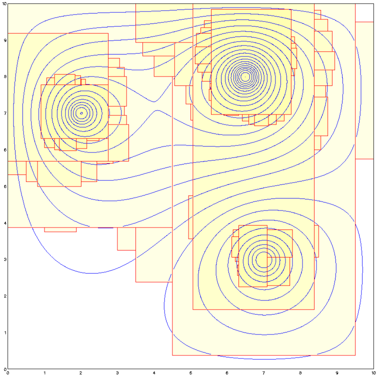

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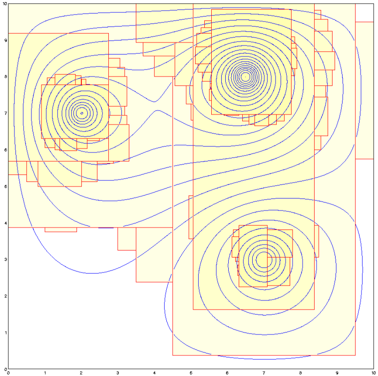

	128 × 128	1024 × 1024	1024 × 1024
l_{max}	3	0	0
l_{min}	-4	-7	-4
ν_1	5	5	5
ν_2	5	5	5
V-Cycles	15	16	341
Time [sec]	9.4	27.7	563

Stop at $\|d^l\|_{max} < 10^{-7}$ for $l \geq 0$, $\gamma = 1$, $r_l = 2$

Some comments on parabolic problems

- Consequences of time step refinement
- Level-wise elliptic solves vs. global solve
- If time step refinement is used an elliptic flux correction is unavoidable.
- The correction is explained in Bell, J. (2004). Block-structured adaptive mesh refinement. Lecture 2. Available at https://ccse.lbl.gov/people/jbb/shortcourse/lecture2.pdf.

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