Lecture 4a

Using the SAMR approach for elliptic and parabolic problems

Course *Block-structured Adaptive Mesh Refinement Methods for Conservation Laws*
Theory, Implementation and Application

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Outline

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Poisson equation

$$\begin{aligned}\Delta q(\mathbf{x}) &= \psi(\mathbf{x})\,, \quad \mathbf{x} \in \Omega \subset \mathbb{R}^d, \quad q \in \mathrm{C}^2(\Omega), \quad \psi \in \mathrm{C}^0(\Omega) \\ q &= \psi^\Gamma(\mathbf{x})\,, \quad \mathbf{x} \in \partial\Omega\end{aligned}$$

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Discrete Poisson equation in 2D:

$$\frac{Q_{j+1,k} - 2Q_{jk} + Q_{j-1,k}}{\Delta x_1^2} + \frac{Q_{j,k+1} - 2Q_{jk} + Q_{j,k-1}}{\Delta x_2^2} = \psi_{jk}$$

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Operator

$$\mathcal{A}(Q_{\Delta x_1, \Delta x_2}) = \begin{bmatrix} & \frac{1}{\Delta x_2^2} & \\ \frac{1}{\Delta x_1^2} & -\left(\frac{2}{\Delta x_1^2} + \frac{2}{\Delta x_2^2}\right) & \frac{1}{\Delta x_2^2} \\ & \frac{1}{\Delta x_2^2} & \end{bmatrix} Q(x_{1,j}, x_{2,k}) = \psi_{jk}$$

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$$Q_{jk} = \frac{1}{\sigma}\left[(Q_{j+1,k} + Q_{j-1,k})\Delta x_2^2 + (Q_{j,k+1} + Q_{j,k-1})\Delta x_1^2 - \Delta x_1^2 \Delta x_2^2 \psi_{jk}\right]$$

with $\sigma = \dfrac{2\Delta x_1^2 + 2\Delta x_2^2}{\Delta x_1^2 \Delta x_2^2}$

Iterative methods

Jacobi iteration

$$Q_{jk}^{m+1} = \frac{1}{\sigma}\left[(Q_{j+1,k}^m + Q_{j-1,k}^m)\Delta x_2^2 + (Q_{j,k+1}^m + Q_{j,k-1}^m)\Delta x_1^2 - \Delta x_1^2 \Delta x_2^2 \psi_{jk}\right]$$

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Lexicographical Gauss-Seidel iteration (use updated values when they become available)

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Efficient parallelization / patch-wise application not possible!

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Checker-board or Red-Black Gauss Seidel iteration

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Gauss-Seidel methods require $\sim 1/2$ of iterations than Jacobi method, however, iteration count still proportional to number of unknowns [Hackbusch, 1994]

Smoothing vs. solving

ν iterations with iterative linear solver

$$Q^{m+\nu} = \mathcal{S}(Q^m, \psi, \nu)$$

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Defect after m iterations

$$d^m = \psi - \mathcal{A}(Q^m)$$

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Defect after $m + \nu$ iterations

$$d^{m+\nu} = \psi - \mathcal{A}(Q^{m+\nu}) = \psi - \mathcal{A}(Q^m + v_\nu^m) = d^m - \mathcal{A}(v_\nu^m)$$

with correction

$$v_\nu^m = \mathcal{S}(\vec{0}, d^m, \nu)$$

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Neglecting the sub-iterations in the smoother we write

$$Q^{n+1} = Q^n + v = Q^n + \mathcal{S}(d^n)$$

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Observation: Oscillations are damped faster on coarser grid.

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Observation: Oscillations are damped faster on coarser grid.

Coarse grid correction:

$$Q^{n+1} = Q^n + v = Q^n + \mathcal{P}\mathcal{S}\mathcal{R}(d^n)$$

where $\mathcal{R}$ is suitable restriction operator and $\mathcal{P}$ a suitable prolongation operator

Two-grid correction method

Relaxation on current grid:

$$\bar{Q} = \mathcal{S}(Q^n, \psi, \nu)$$

$$Q^{n+1} = \bar{Q} + \mathcal{P}\mathcal{S}(\vec{0}, \cdot, \mu)\mathcal{R}(\psi - \mathcal{A}(\bar{Q}))$$

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Algorithm:

$\bar{Q} := \mathcal{S}(Q^n, \psi, \nu)$
$d := \psi - \mathcal{A}(\bar{Q})$

$d_c := \mathcal{R}(d)$
$v_c := \mathcal{S}(0, d_c, \mu)$
$v := \mathcal{P}(v_c)$
$Q^{n+1} := \bar{Q} + v$

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Algorithm:	with smoothing:
$\bar{Q} := \mathcal{S}(Q^n, \psi, \nu)$	$d := \psi - \mathcal{A}(Q)$
$d := \psi - \mathcal{A}(\bar{Q})$	$v := \mathcal{S}(0, d, \nu)$
	$r := d - \mathcal{A}(v)$
$d_c := \mathcal{R}(d)$	$d_c := \mathcal{R}(r)$
$v_c := \mathcal{S}(0, d_c, \mu)$	$v_c := \mathcal{S}(0, d_c, \mu)$
$v := \mathcal{P}(v_c)$	$v := v + \mathcal{P}(v_c)$
$Q^{n+1} := \bar{Q} + v$	$Q^{n+1} := Q + v$

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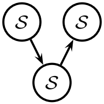
with pre- and post-iteration:

$$\begin{aligned} d &:= \psi - \mathcal{A}(Q) \\ v &:= \mathcal{S}(0, d, \nu_1) \\ r &:= d - \mathcal{A}(v) \\ d_c &:= \mathcal{R}(r) \\ v_c &:= \mathcal{S}(0, d_c, \mu) \\ v &:= v + \mathcal{P}(v_c) \\ d &:= d - \mathcal{A}(v) \\ r &:= \mathcal{S}(0, d, \nu_2) \\ Q^{n+1} &:= Q + v + r \end{aligned}$$

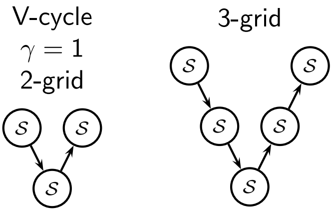
[Hackbusch, 1985]

Multi-level methods and cycles

V-cycle
$\gamma = 1$
2-grid



Multi-level methods and cycles



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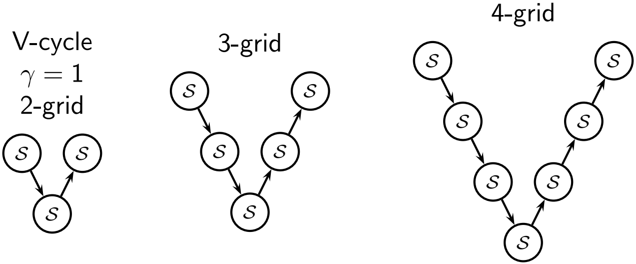

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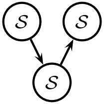
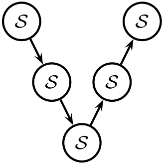
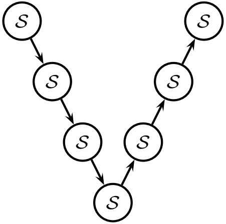


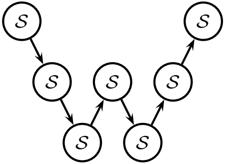
3-grid



4-grid



W-cycle
$\gamma = 2$



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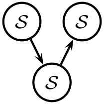

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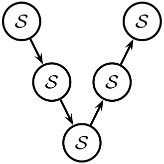

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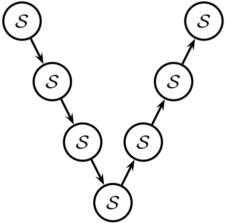

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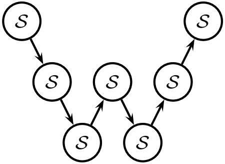

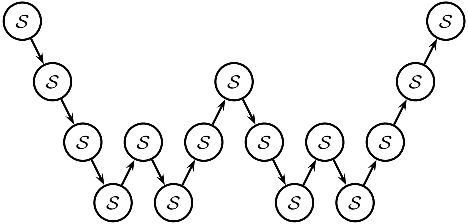

[Hackbusch, 1985] [Wesseling, 1992] ...

Stencil modification at coarse-fine boundaries in 1D

1D Example: Cell j, $\psi - \nabla \cdot \nabla q = 0$

$$d_j^l = \psi_j - \frac{1}{\Delta x_l}\left(\frac{1}{\Delta x_l}(Q_{j+1}^l - Q_j^l) - \frac{1}{\Delta x_l}(Q_j^l - Q_{j-1}^l)\right)$$

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H is approximation to *derivative* of Q^l.

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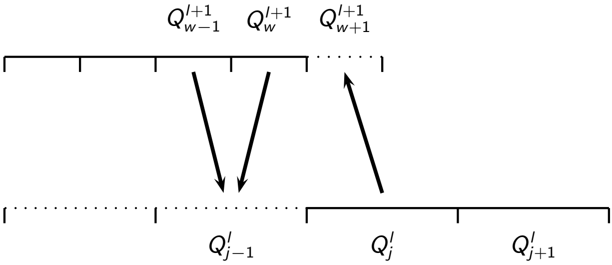

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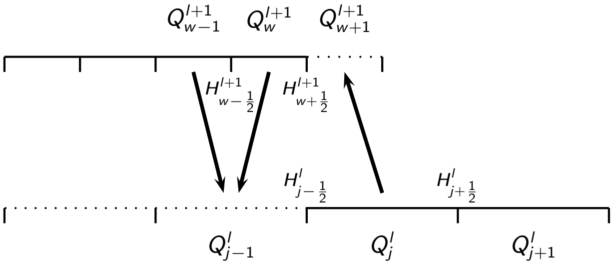

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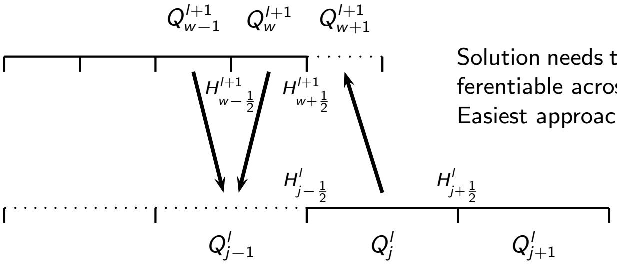

Solution needs to be continuously differentiable across interface.
Easiest approach: $H_{w+\frac{1}{2}}^{l+1} \equiv H_{j-\frac{1}{2}}^l$

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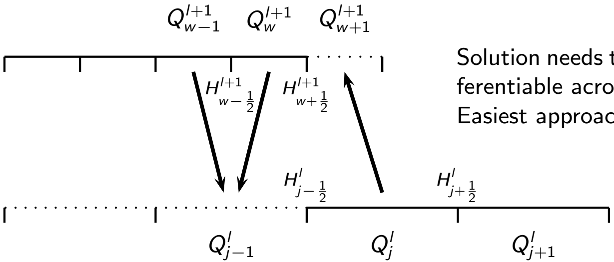

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No specific modification necessary for 1D vertex-based stencils, cf. [Bastian, 1996]

Stencil modification at coarse-fine boundaries in 1D II

Set $H^{l+1}_{w+\frac{1}{2}} = H_{\mathcal{I}}$.

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$$\frac{Q^{l+1}_{w+1} - Q^{l+1}_{w}}{\Delta x_{l+1}} = \frac{Q^{l}_{j} - Q^{l+1}_{w}}{\frac{3}{2}\Delta x_{l+1}}$$

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from which we readily derive

$$Q^{l+1}_{w+1} = \frac{2}{3} Q^{l}_{j} + \frac{1}{3} Q^{l+1}_{w}$$

for the boundary cell on $l+1$.

Stencil modification at coarse-fine boundaries in 1D II

Set $H^{l+1}_{w+\frac{1}{2}} = H_{\mathcal{I}}$. Inserting Q gives

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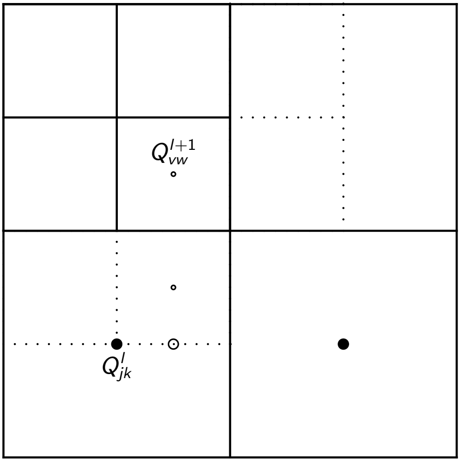
yields

$$\check{d}^{l}_{j} = \psi_j - \frac{1}{\Delta x_l}\left(\frac{1}{\Delta x_l}(Q^{l}_{j+1} - Q^{l}_{j}) - \frac{2}{3\Delta x_{l+1}}(Q^{l}_{j} - Q^{l+1}_{w})\right)$$

Stencil modification at coarse-fine boundaries: 2D



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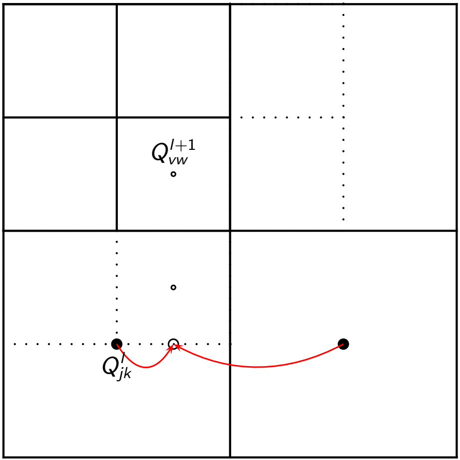

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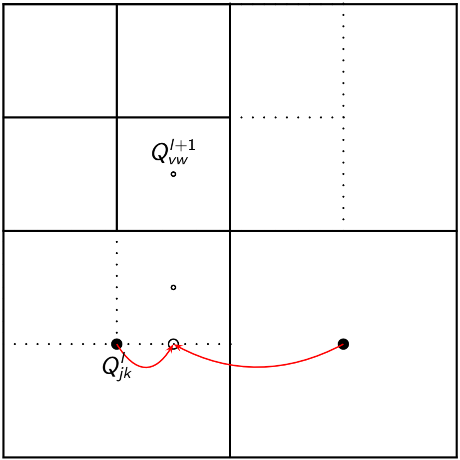

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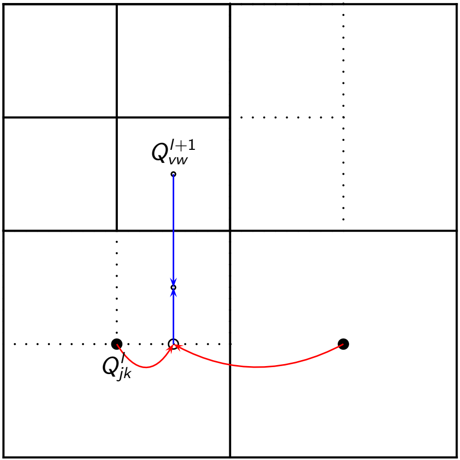

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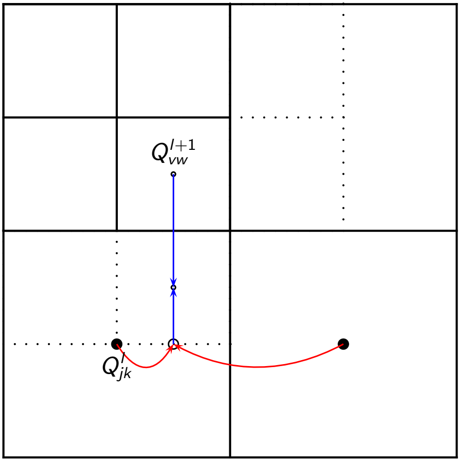

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In general:

$$Q_{v,w-1}^{l+1} = \left(1 - \frac{2}{r_{l+1}+1}\right) Q_{vw}^{l+1} + \frac{2}{r_{l+1}+1}\left((1-f)Q_{jk}^{l} + fQ_{j+1,k}^{l}\right)$$

with

$$f = \frac{x_{1,l+1}^{v} - x_{1,l}^{j}}{\Delta x_{1,l}}$$

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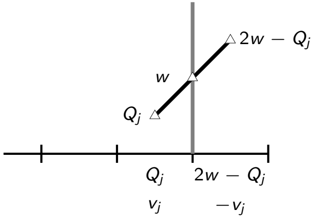

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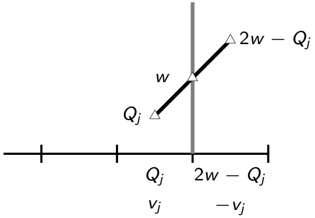

- Coarse-fine boundary flux accumulation and application of δH^{l+1} on defect d^l

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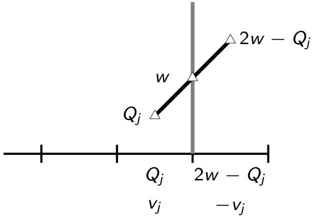

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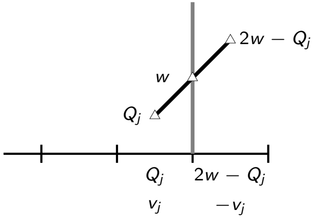

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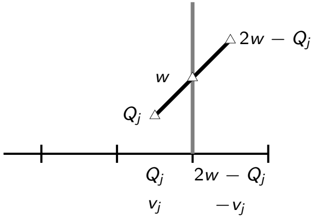

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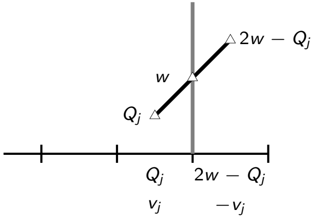

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- Looping instead of time steps and check of convergence

Additive geometric multigrid algorithm

```
AdvanceLevelMG(l) - Correction Scheme

 Set ghost cells of Q^l
 Calculate defect d^l from Q^l, ψ^l                      d^l := ψ^l − A(Q^l)
 If (l < l_max)
     Calculate updated defect r^{l+1} from v^{l+1}, d^{l+1}     r^{l+1} := d^{l+1} − A(v^{l+1})
     Restrict d^{l+1} onto d^l                                 d^l := R_l^{l+1}(r^{l+1})
 Do ν_1 smoothing steps to get correction v^l            v^l := S(0, d^l, ν_1)
 If (l > l_min)
     Do γ > 1 times
         AdvanceLevelMG(l − 1)
     Set ghost cells of v^{l−1}
     Prolongate and add v^{l−1} to v^l                          v^l := v^l + P_l^{l−1}(v^{l−1})
     If (ν_2 > 0)
         Set ghost cells of v^l
         Update defect d^l according to v^l                        d^l := d^l − A(v^l)
         Do ν_2 post-smoothing steps to get r^l                    r^l := S(v^l, d^l, ν_2)
         Add addional correction r^l to v^l                        v^l := v^l + r^l
 Add correction v^l to Q^l                               Q^l := Q^l + v^l
```

Additive Geometric Multiplicative Multigrid Algorithm

Start - Start iteration on level l_{max}

For $l = l_{max}$ Downto $l_{min} + 1$ Do
    Restrict Q^l onto Q^{l-1}
Regrid(0)
AdvanceLevelMG(l_{max})

See also: [Trottenberg et al., 2001], [Canu and Ritzdorf, 1994]
Vertex-based: [Brandt, 1977], [Briggs et al., 2001]

Example

On $\Omega = [0,10] \times [0,10]$ use hat function

$$\psi = \begin{cases} -A_n \cos\left(\frac{\pi r}{2R_n}\right) \;, & r < R_n \\ 0 & \text{elsewhere} \end{cases}$$

with $r = \sqrt{(x_1 - X_n)^2 + (x_2 - Y_n)^2}$ to define three sources with

n	A_n	R_n	X_n	Y_n
1	0.3	0.3	6.5	8.0
2	0.2	0.3	2.0	7.0
3	-0.1	0.4	7.0	3.0

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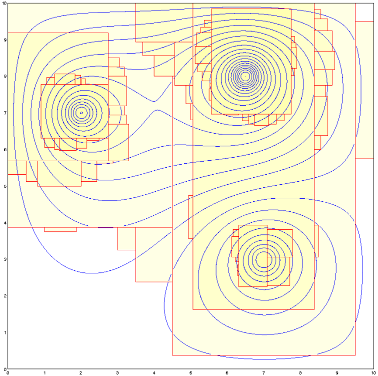

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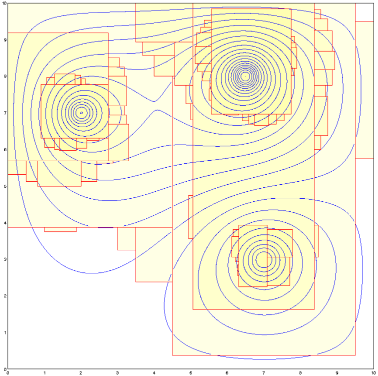

	128×128	1024×1024	1024×1024
l_{max}	3	0	0
l_{min}	-4	-7	-4
ν_1	5	5	5
ν_2	5	5	5
V-Cycles	15	16	341
Time [sec]	9.4	27.7	563

Stop at $\|d^l\|_{max} < 10^{-7}$ for $l \geq 0,\ \gamma = 1,\ r_l = 2$

Some comments on parabolic problems

- Consequences of time step refinement
- Level-wise elliptic solves vs. global solve
- If time step refinement is used an elliptic flux correction is unavoidable.
- The correction is explained in Bell, J. (2004). Block-structured adaptive mesh refinement. Lecture 2. Available at https://ccse.lbl.gov/people/jbb/shortcourse/lecture2.pdf.

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