Lecture 1

Fundamentals: Used schemes and mesh adaptation

Course *Block-structured Adaptive Mesh Refinement Methods for Conservation Laws*
Theory, Implementation and Application

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Outline

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Hyperbolic Conservation Laws

$$\frac{\partial}{\partial t}\mathbf{q}(\mathbf{x},t) + \sum_{n=1}^{d} \frac{\partial}{\partial x_n}\mathbf{f}_n(\mathbf{q}(\mathbf{x},t)) = 0\,, \quad D \subset \{(\mathbf{x},t) \in \mathbb{R}^d \times \mathbb{R}_0^+\}$$

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Definition (Hyperbolicity)

$\mathbf{A}(\mathbf{q},\nu) = \nu_1\mathbf{A}_1(\mathbf{q}) + \cdots + \nu_d\mathbf{A}_d(\mathbf{q})$ with $\mathbf{A}_n(\mathbf{q}) = \partial\mathbf{f}_n(\mathbf{q})/\partial\mathbf{q}$ has M real eigenvalues $\lambda_1(\mathbf{q},\nu) \le \ldots \le \lambda_M(\mathbf{q},\nu)$ and M linear independent right eigenvectors $\mathbf{r}_m(\mathbf{q},\nu)$.

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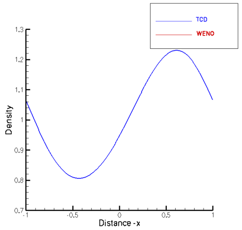
If $\mathbf{f}_n(\mathbf{q})$ is nonlinear, classical solutions $\mathbf{q}(\mathbf{x},t) \in \mathrm{C}^1(D,S)$ do not generally exist, not even for $\mathbf{q}_0(\mathbf{x}) \in \mathrm{C}^1(\mathbb{R}^d,S)$ [Majda, 1984], [Godlewski and Raviart, 1996], [Kröner, 1997]



Example: Euler equations

Weak solutions

Integral form (Gauss's theorem):

$$\int_{\Omega} \mathbf{q}(\mathbf{x}, t+\Delta t)\, d\mathbf{x} - \int_{\Omega} \mathbf{q}(\mathbf{x}, t)\, d\mathbf{x}$$

$$+ \sum_{n=1}^{d} \int_{t}^{t+\Delta t} \int_{\partial\Omega} \mathbf{f}_n(\mathbf{q}(\mathbf{o}, t))\, \sigma_n(\mathbf{o})\, d\mathbf{o}\, dt = \int_{t}^{t+\Delta t} \int_{\Omega} \mathbf{s}(\mathbf{q}(\mathbf{x}, t))\, d\mathbf{x}$$

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Theorem (Weak solution)

$\mathbf{q}_0 \in \mathrm{L}^{\infty}_{loc}(\mathbb{R}^d, S)$. $\mathbf{q} \in \mathrm{L}^{\infty}_{loc}(D, S)$ *is weak solution if* $\mathbf{q}$ *satisfies*

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$$\int_0^\infty \int_{\mathbb{R}^d} \left[\frac{\partial \varphi}{\partial t} \cdot \mathbf{q} + \sum_{n=1}^{d} \frac{\partial \varphi}{\partial x_n} \cdot \mathbf{f}_n(\mathbf{q}) - \varphi \cdot \mathbf{s}(\mathbf{q}) \right] d\mathbf{x}\, dt + \int_{\mathbb{R}^d} \varphi(\mathbf{x}, 0) \cdot \mathbf{q}_0(\mathbf{x})\, d\mathbf{x} = 0$$

for any test function $\varphi \in \mathrm{C}^1_0(D, S)$

Entropy solutions

Select physical weak solution as $\lim_{\varepsilon \longrightarrow 0} \mathbf{q}_\varepsilon = \mathbf{q}$ almost everywhere in D of

$$\frac{\partial \mathbf{q}_\varepsilon}{\partial t} + \sum_{n=1}^{d} \frac{\partial \mathbf{f}_n(\mathbf{q}_\varepsilon)}{\partial x_n} - \varepsilon \sum_{n=1}^{d} \frac{\partial^2 \mathbf{q}_\varepsilon}{\partial x_n^2} = \mathbf{s}(\mathbf{q}_\varepsilon)\,, \quad \mathbf{x} \in \mathbb{R}^d\,, \quad t > 0$$

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then $\lim_{\varepsilon\longrightarrow 0} \mathbf{q}_\varepsilon = \mathbf{q}$ *almost everywhere in* D *is weak solution and satisfies*

$$\frac{\partial \eta(\mathbf{q})}{\partial t} + \sum_{n=1}^{d} \frac{\partial \psi_n(\mathbf{q})}{\partial x_n} \le \frac{\partial \eta(\mathbf{q})}{\partial \mathbf{q}}^T \cdot \mathbf{s}(\mathbf{q})$$

in the sense of distributions. Proof: [Godlewski and Raviart, 1996]

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for all entropy functions $\eta(\mathbf{q})$ and all test functions $\varphi \in \mathrm{C}_0^1(D, \mathbb{R}_0^+)$, $\varphi \geq 0$

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An entropy solution $\mathbf{q}$ *is a classical solution* $\mathbf{q} \in \mathrm{C}^1(D,S)$ *almost everywhere and satisfies the Rankine-Hugoniot (RH) jump condition*

$$(\mathbf{q}^+ - \mathbf{q}^-)\, \sigma_t + \sum_{n=1}^{d} \left(\mathbf{f}_n(\mathbf{q}^+) - \mathbf{f}_n(\mathbf{q}^-)\right) \sigma_n = \mathbf{0}$$

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and the jump inequality

$$(\eta(\mathbf{q}^+) - \eta(\mathbf{q}^-))\, \sigma_t + \sum_{n=1}^d \left(\psi_n(\mathbf{q}^+) - \psi_n(\mathbf{q}^-) \right) \sigma_n \leq 0$$

along discontinuities. Proof: [Godlewski and Raviart, 1996]

Examples

Euler equations

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_n}\left(\rho u_n\right) = 0$$

$$\frac{\partial}{\partial t}\left(\rho u_k\right) + \frac{\partial}{\partial x_n}\left(\rho u_k u_n + \delta_{kn} p\right) = 0\ , \quad k = 1, \ldots, d$$

$$\frac{\partial}{\partial t}\left(\rho E\right) + \frac{\partial}{\partial x_n}\left(u_n(\rho E + p)\right) = 0$$

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$$p = (\gamma - 1)\big(\rho E - \frac{1}{2}\rho u_n u_n\big)$$

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Navier-Stokes equations

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with stress tensor

$$\tau_{kn} = \mu\Big(\frac{\partial u_n}{\partial x_k} + \frac{\partial u_k}{\partial x_n}\Big) - \frac{2}{3}\mu \frac{\partial u_j}{\partial x_j}\delta_{kn}$$

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$$q_n = -\lambda \frac{\partial T}{\partial x_n}$$

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Type can be either hyperbolic or parabolic

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Time discretization $t_n = n\Delta t$, discrete volumes
$I_j = [x_j - \frac{1}{2}\Delta x, x_j + \frac{1}{2}\Delta x[=: [x_{j-1/2}, x_{j+1/2}[$

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Using approximations $\mathbf{Q}_j(t) \approx \frac{1}{|I_j|} \int\limits_{I_j} \mathbf{q}(\mathbf{x}, t)\, dx, \quad \mathbf{s}(\mathbf{Q}_j(t)) \approx \frac{1}{|I_j|} \int\limits_{I_j} \mathbf{s}(\mathbf{q}(\mathbf{x}, t))\, dx$

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yields after integration (Gauss theorem)

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$$\frac{1}{\Delta x} \int_{t_n}^{t_{n+1}} \left[\mathbf{H}\left(\mathbf{Q}_j(t), \mathbf{Q}_{j+1}(t)\right) - \mathbf{H}\left(\mathbf{Q}_{j-1}(t), \mathbf{Q}_j(t)\right)\right] dt + \int_{t_n}^{t_{n+1}} \mathbf{s}(\mathbf{Q}_j(t))\, dt$$

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For instance:

$$\mathbf{Q}_j^{n+1} = \mathbf{Q}_j^n - \frac{\Delta t}{\Delta x}\left[\mathbf{F}\left(\mathbf{Q}_j^n, \mathbf{Q}_{j+1}^n\right) - \mathbf{F}\left(\mathbf{Q}_{j-1}^n, \mathbf{Q}_j^n\right)\right] -$$
$$\frac{\Delta t}{\Delta x}\left[\mathbf{H}\left(\mathbf{Q}_j^n, \mathbf{Q}_{j+1}^n\right) - \mathbf{H}\left(\mathbf{Q}_{j-1}^n, \mathbf{Q}_j^n\right)\right] + \Delta t \mathbf{s}(\mathbf{Q}_j^n)\, dt$$

Some classical definitions

$(2s+1)$-point difference scheme of the form

$$\mathbf{Q}_j^{n+1} = \mathcal{H}^{(\Delta t)}(\mathbf{Q}_{j-s}^n, \ldots, \mathbf{Q}_{j+s}^n)$$

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Definition (Stability)

For each time τ there is a constant C_S and a value $n_0 \in \mathbb{N}$ such that $\|\mathcal{H}^{(\Delta t)}(\mathbf{Q}^n)\| \leq C_S$ for all $n\Delta t \leq \tau$, $n < n_0$

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If the local truncation error

$$\mathcal{L}^{(\Delta t)}(\mathbf{x}, t) := \frac{1}{\Delta t}\left[\mathbf{q}(\mathbf{x}, t+\Delta t) - \mathcal{H}^{(\Delta t)}(\mathbf{q}(\cdot, t))\right]$$

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Some classical definitions II

Definition (Order of accuracy)

$\mathcal{H}(\cdot)$ is accurate of order o if for all sufficiently smooth initial data $\mathbf{q}_0(\mathbf{x})$, there is a constant C_L, such that the local truncation error satisfies
$\|\mathcal{L}^{(\Delta t)}(\cdot, t)\| \le C_L \Delta t^o$ for all $\Delta t < \Delta t_0$, $t \le \tau$

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Splitting methods

Solve homogeneous PDE and ODE successively!

$$\begin{aligned} \mathcal{H}^{(\Delta t)} : &\quad \partial_t \mathbf{q} + \nabla \cdot \mathbf{f}(\mathbf{q}) = 0 \,, &\quad \text{IC: } \mathbf{Q}(t_m) \stackrel{\Delta t}{\Longrightarrow} \tilde{\mathbf{Q}} \\ \mathcal{S}^{(\Delta t)} : &\quad \partial_t \mathbf{q} = \mathbf{s}(\mathbf{q}) \,, &\quad \text{IC: } \tilde{\mathbf{Q}} \stackrel{\Delta t}{\Longrightarrow} \mathbf{Q}(t_m + \Delta t) \end{aligned}$$

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1st-order dimensional splitting for $\mathcal{H}^{(\cdot)}$:

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[Toro, 1999]

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i.e.

$$|1 - 4C_1 - 4C_2| \le 1$$

from which we derive the stability condition

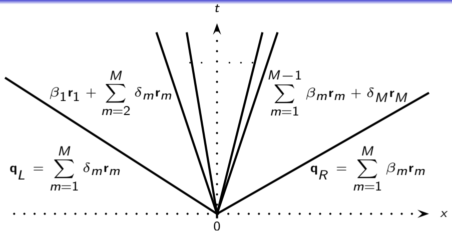
$$0 \le c\left(\frac{\Delta t}{\Delta x_1^2} + \frac{\Delta t}{\Delta x_2^2}\right) \le \frac{1}{2}$$

Linear upwind schemes

Consider Riemann problem

$$\frac{\partial}{\partial t}\mathbf{q}(x,t)+\mathbf{A}\frac{\partial}{\partial x}\mathbf{q}(x,t)=\mathbf{0}\,, \quad x\in\mathbb{R}\,, \quad t>0$$

Linear upwind schemes



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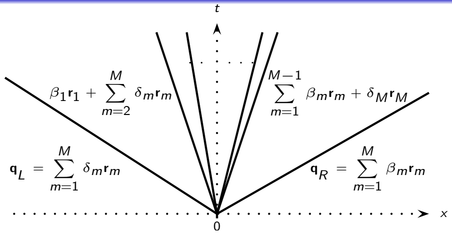

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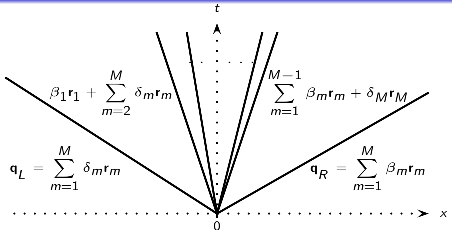

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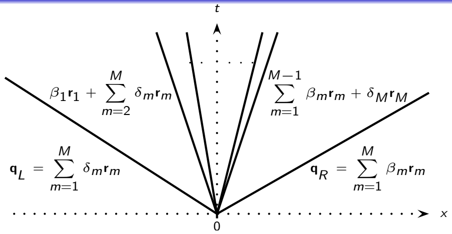

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and $\quad \mathbf{A}^+ := \mathbf{R}\,\mathbf{\Lambda}^+\,\mathbf{R}^{-1}, \qquad \mathbf{A}^- := \mathbf{R}\,\mathbf{\Lambda}^-\,\mathbf{R}^{-1} \qquad$ which gives

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with $\Delta\mathbf{q} = \mathbf{q}_R - \mathbf{q}_L$

Flux difference splitting

Godunov-type scheme with $\Delta\mathbf{Q}^n_{j+1/2} = \mathbf{Q}^n_{j+1} - \mathbf{Q}^n_j$

$$\mathbf{Q}^{n+1}_j = \mathbf{Q}^n_j - \frac{\Delta t}{\Delta x}\left(\mathbf{A}^-\Delta\mathbf{Q}^n_{j+1/2} + \mathbf{A}^+\Delta\mathbf{Q}^n_{j-1/2}\right)$$

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Use linearization $\bar{\mathbf{f}}(\bar{\mathbf{q}}) = \hat{\mathbf{A}}(\mathbf{q}_L, \mathbf{q}_R)\bar{\mathbf{q}}$ and construct scheme for nonlinear problem as

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Flux difference splitting

Godunov-type scheme with $\Delta \mathbf{Q}^n_{j+1/2} = \mathbf{Q}^n_{j+1} - \mathbf{Q}^n_j$

$$\mathbf{Q}^{n+1}_j = \mathbf{Q}^n_j - \frac{\Delta t}{\Delta x}\left(\mathbf{A}^- \Delta \mathbf{Q}^n_{j+1/2} + \mathbf{A}^+ \Delta \mathbf{Q}^n_{j-1/2}\right)$$

Use linearization $\bar{\mathbf{f}}(\bar{\mathbf{q}}) = \hat{\mathbf{A}}(\mathbf{q}_L, \mathbf{q}_R)\bar{\mathbf{q}}$ and construct scheme for nonlinear problem as

$$\mathbf{Q}^{n+1}_j = \mathbf{Q}^n_j - \frac{\Delta t}{\Delta x}\left(\hat{\mathbf{A}}^-(\mathbf{Q}^n_j, \mathbf{Q}^n_{j+1})\Delta \mathbf{Q}^n_{j+\frac{1}{2}} + \hat{\mathbf{A}}^+(\mathbf{Q}^n_{j-1}, \mathbf{Q}^n_j)\Delta \mathbf{Q}^n_{j-\frac{1}{2}}\right)$$

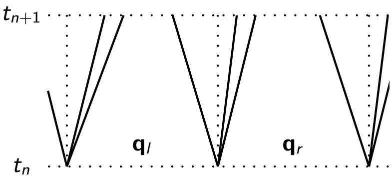
stability condition

$$\max_{j\in\mathbb{Z}} |\hat{\lambda}_{m,j+\frac{1}{2}}| \frac{\Delta t}{\Delta x} \le 1 \ , \quad \text{for all } m = 1, \ldots, M$$

[LeVeque, 1992]

Roe's approximate Riemann solver

Choosing $\hat{\mathbf{A}}(\mathbf{q}_L, \mathbf{q}_R)$ [Roe, 1981]:



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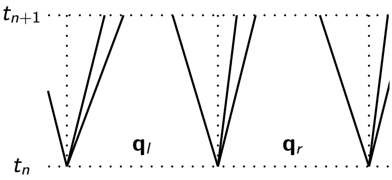

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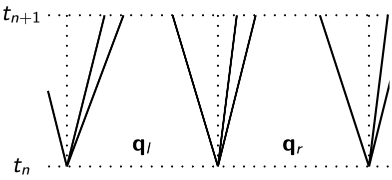

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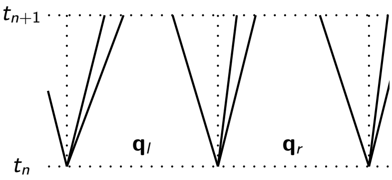

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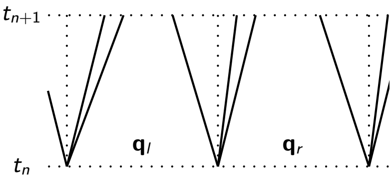

For Euler equations:

$$\hat{\rho} = \frac{\sqrt{\rho_L}\rho_R + \sqrt{\rho_R}\rho_L}{\sqrt{\rho_L} + \sqrt{\rho_R}} = \sqrt{\rho_L \rho_R} \quad \text{and} \quad \hat{v} = \frac{\sqrt{\rho_L} v_L + \sqrt{\rho_R} v_R}{\sqrt{\rho_L} + \sqrt{\rho_R}} \quad \text{for } v = u_n, H$$

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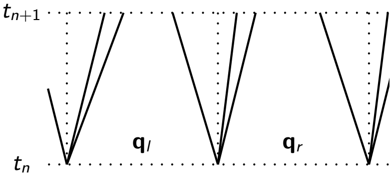

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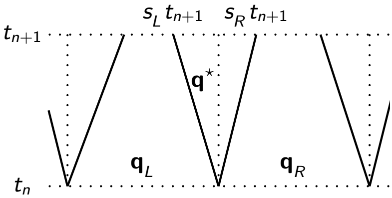
Wave decomposition: $\Delta\mathbf{q} = \mathbf{q}_r - \mathbf{q}_l = \sum_m a_m \, \hat{\mathbf{r}}_m$

$$\begin{aligned} \mathbf{F}(\mathbf{q}_L, \mathbf{q}_R) &= \mathbf{f}(\mathbf{q}_L) + \sum_{\hat{\lambda}_m < 0} \hat{\lambda}_m \, a_m \, \hat{\mathbf{r}}_m = \mathbf{f}(\mathbf{q}_R) - \sum_{\hat{\lambda}_m \geq 0} \hat{\lambda}_m \, a_m \, \hat{\mathbf{r}}_m \\ &= \frac{1}{2}\left(\mathbf{f}(\mathbf{q}_L) + \mathbf{f}(\mathbf{q}_R) - \sum_m |\hat{\lambda}_m| \, a_m \, \hat{\mathbf{r}}_m \right) \end{aligned}$$

Harten-Lax-Van Leer (HLL) approximate Riemann solver



$$\bar{\mathbf{q}}(x,t) = \begin{cases} \mathbf{q}_L\,, & x < s_L\, t \\ \mathbf{q}^\star\,, & s_L\, t \le x \le s_R\, t \\ \mathbf{q}_R\,, & x > s_R\, t \end{cases}$$

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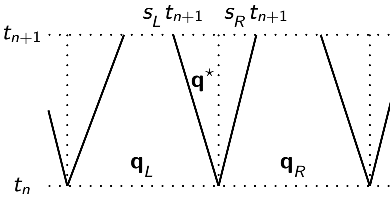

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$$\mathbf{F}_{HLL}(\mathbf{q}_L, \mathbf{q}_R) = \begin{cases} \mathbf{f}(\mathbf{q}_L)\,, & 0 < s_L\,, \\ \dfrac{s_R \mathbf{f}(\mathbf{q}_L) - s_L \mathbf{f}(\mathbf{q}_R) + s_L s_R (\mathbf{q}_R - \mathbf{q}_L)}{s_R - s_L}\,, & s_L \le 0 \le s_R\,, \\ \mathbf{f}(\mathbf{q}_R)\,, & 0 > s_R\,, \end{cases}$$

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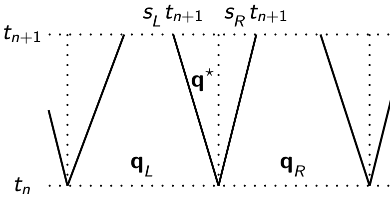

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Euler equations:

$$s_L = \min(u_{1,L} - c_L, u_{1,R} - c_R)\,, \quad s_R = \max(u_{1,L} + c_l, u_{1,R} + c_R)$$

[Toro, 1999], HLLC: [Toro et al., 1994]

Flux vector splitting

Splitting

$$\mathbf{f}(\mathbf{q}) = \mathbf{f}^+(\mathbf{q}) + \mathbf{f}^-(\mathbf{q})$$

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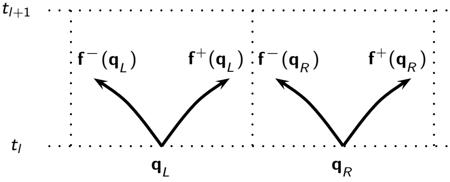

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$$\mathbf{f}(\mathbf{q}) = \mathbf{f}^+(\mathbf{q}) + \mathbf{f}^-(\mathbf{q})$$

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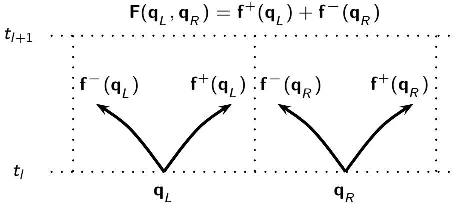

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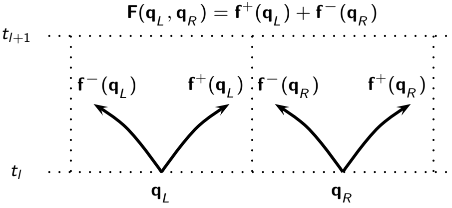

plus reproduction of regular upwinding

$$\begin{array}{llllll} \mathbf{f}^{+}(\mathbf{q}) = \mathbf{f}(\mathbf{q}), & \mathbf{f}^{-}(\mathbf{q}) = \mathbf{0} & \text{if} & \lambda_m \geq 0 & \text{for all} & m = 1, \dots, M \\ \mathbf{f}^{+}(\mathbf{q}) = \mathbf{0}, & \mathbf{f}^{-}(\mathbf{q}) = \mathbf{f}(\mathbf{q}) & \text{if} & \lambda_m \leq 0 & \text{for all} & m = 1, \dots, M \end{array}$$

Then use

$$\mathbf{F}(\mathbf{q}_L, \mathbf{q}_R) = \mathbf{f}^{+}(\mathbf{q}_L) + \mathbf{f}^{-}(\mathbf{q}_R)$$

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Jacobians of the split fluxes are identical to $\mathbf{A}^\pm(\mathbf{q})$ only in linear case

$$\frac{\partial \mathbf{f}^\pm(\mathbf{q})}{\partial \mathbf{q}} = \frac{\partial\left(\mathbf{A}^\pm(\mathbf{q})\,\mathbf{q}\right)}{\partial \mathbf{q}} = \mathbf{A}^\pm(\mathbf{q}) + \frac{\partial \mathbf{A}^\pm(\mathbf{q})}{\partial \mathbf{q}}\,\mathbf{q}$$

Further methods: Van Leer FVS [Toro, 1999], AUSM [Wada and Liou, 1997]

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Objective: Higher-order accuracy in smooth solution regions but no spurious oscillations near large gradients

Consistent monotone methods converge toward the entropy solution, but

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Definition (TVD property)

Scheme $\mathcal{H}^{(\Delta t)}(\mathbf{Q}^n;j)$ TVD if $TV(\mathbf{Q}^{l+1}) \le TV(\mathbf{Q}^l)$ is satisfied for all discrete sequences $\mathbf{Q}^n$. Herein, $TV(\mathbf{Q}^l) := \sum_{j\in\mathbb{Z}} |\mathbf{Q}^l_{j+1} - \mathbf{Q}^l_j|$.

TVD schemes: no new extrema, local minima are non-decreasing, local maxima are non-increasing (termed *monotonicity-preserving*). *Monotonicity-preserving* higher-order schemes are at least 5-point methods. Proofs: [Harten, 1983]

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TVD concept is proven [Godlewski and Raviart, 1996] for scalar schemes only but nevertheless used to construct *high resolution* schemes.
Monotonicity-preserving scheme can converge toward non-physical weak solutions.

MUSCL slope limiting

Monotone Upwind Schemes for Conservation Laws [van Leer, 1979]

$$\tilde{Q}^{L}_{j+\frac{1}{2}} = Q^n_j + \frac{1}{4}\left[(1-\omega)\,\Phi^{+}_{j-\frac{1}{2}}\Delta_{j-\frac{1}{2}} + (1+\omega)\,\Phi^{-}_{j+\frac{1}{2}}\Delta_{j+\frac{1}{2}}\right] ,$$

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$$\Phi^{+}_{j-\frac{1}{2}} := \Phi\left(r^{+}_{j-\frac{1}{2}}\right), \quad \Phi^{-}_{j+\frac{1}{2}} := \Phi\left(r^{-}_{j+\frac{1}{2}}\right) \quad \text{with} \quad r^{+}_{j-\frac{1}{2}} := \frac{\Delta_{j+\frac{1}{2}}}{\Delta_{j-\frac{1}{2}}}, \quad r^{-}_{j+\frac{1}{2}} := \frac{\Delta_{j-\frac{1}{2}}}{\Delta_{j+\frac{1}{2}}}$$

and *slope limiters*, e.g., *Minmod*

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Using a midpoint rule for temporal integration, e.g.,

$$Q^{\star}_j = Q^n_j - \frac{1}{2}\frac{\Delta t}{\Delta x}\left(F(Q^n_{j+1}, Q^n_j) - F(Q^n_j, Q^n_{j-1})\right)$$

and constructing limited values from $Q^{\star}$ to be used in FV scheme gives a TVD method if

$$\frac{1}{2}\left[(1-\omega)\Phi(r) + (1+\omega)\, r\, \Phi\left(\frac{1}{r}\right)\right] < \min(2, 2r)$$

is satisfied for $r > 0$. Proof: [Hirsch, 1988]

Wave Propagation with flux limiting

Wave Propagation Method [LeVeque, 1997] is built on the flux differencing approach $\mathcal{A}^{\pm}\Delta := \hat{\mathbf{A}}^{\pm}(\mathbf{q}_L, \mathbf{q}_R)\Delta\mathbf{q}$ and the *waves* $\mathcal{W}_m := a_m \hat{\mathbf{r}}_m$, i.e.

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$$\mathbf{Q}^{n+1} = \mathbf{Q}_j^n - \frac{\Delta t}{\Delta x}\left(\mathcal{A}^-\Delta_{j+\frac{1}{2}} + \mathcal{A}^+\Delta_{j-\frac{1}{2}}\right) - \frac{\Delta t}{\Delta x}\left(\tilde{\mathbf{F}}_{j+\frac{1}{2}} - \tilde{\mathbf{F}}_{j-\frac{1}{2}}\right)$$

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and wave limiter

$$\tilde{\mathcal{W}}^m_{j+\frac{1}{2}} = \Phi(\Theta^m_{j+\frac{1}{2}})\,\mathcal{W}^m_{j+\frac{1}{2}}$$

with

$$\Theta^m_{j+\frac{1}{2}} = \begin{cases} a^m_{j-\frac{1}{2}}/a^m_{j+\frac{1}{2}}\,, & \hat{\lambda}^m_{j+\frac{1}{2}} \ge 0\,, \\ a^m_{j+\frac{3}{2}}/a^m_{j+\frac{1}{2}}\,, & \hat{\lambda}^m_{j+\frac{1}{2}} < 0 \end{cases}$$

Wave Propagation Method in 2D

Writing $\tilde{\mathcal{A}}^{\pm}\Delta_{j\pm1/2} := \mathcal{A}^{+}\Delta_{j\pm1/2} + \tilde{\mathbf{F}}_{j\pm1/2}$ one can develop a truly two-dimensional one-step method [Langseth and LeVeque, 2000]

$$\begin{aligned}\mathbf{Q}_{jk}^{n+1} = \mathbf{Q}_{jk}^{n} - \frac{\Delta t}{\Delta x_1}\Bigg(&\tilde{\mathcal{A}}^{-}\Delta_{j+\frac12,k} - \frac12\frac{\Delta t}{\Delta x_2}\left[\mathcal{A}^{-}\tilde{\mathcal{B}}^{-}\Delta_{j+1,k+\frac12} + \mathcal{A}^{-}\tilde{\mathcal{B}}^{+}\Delta_{j+1,k-\frac12}\right] + \\ &\tilde{\mathcal{A}}^{+}\Delta_{j-\frac12,k} - \frac12\frac{\Delta t}{\Delta x_2}\left[\mathcal{A}^{+}\tilde{\mathcal{B}}^{-}\Delta_{j-1,k+\frac12} + \mathcal{A}^{+}\tilde{\mathcal{B}}^{+}\Delta_{j-1,k-\frac12}\right]\Bigg) \\ - \frac{\Delta t}{\Delta x_2}\Bigg(&\tilde{\mathcal{B}}^{-}\Delta_{j,k+\frac12} - \frac12\frac{\Delta t}{\Delta x_1}\left[\mathcal{B}^{-}\tilde{\mathcal{A}}^{-}\Delta_{j+\frac12,k+1} + \mathcal{B}^{-}\tilde{\mathcal{A}}^{+}\Delta_{j-\frac12,k+1}\right] + \\ &\tilde{\mathcal{B}}^{+}\Delta_{j,k-\frac12} - \frac12\frac{\Delta t}{\Delta x_1}\left[\mathcal{B}^{+}\tilde{\mathcal{A}}^{-}\Delta_{j+\frac12,k-1} + \mathcal{B}^{+}\tilde{\mathcal{A}}^{+}\Delta_{j-\frac12,k-1}\right]\Bigg)\end{aligned}$$

that is stable for

$$\left\{\max_{j\in\mathbb{Z}}|\hat{\lambda}_{m,j+\frac12}|\frac{\Delta t}{\Delta x_1}, \max_{k\in\mathbb{Z}}|\hat{\lambda}_{m,k+\frac12}|\frac{\Delta t}{\Delta x_2}\right\} \le 1\,, \quad \text{for all } m = 1,\ldots,M$$

Further high-resolution methods

Some further high-resolution methods (good overview in [Laney, 1998]):

- FCT: 2nd order [Oran and Boris, 2001]

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3rd order methods must make use of strong-stability preserving Runge-Kutta methods [Gottlieb et al., 2001] for time integration that use a multi-step update

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with $\tilde{\mathbf{Q}}^0 := \mathbf{Q}^n$, $\alpha_1 = 1$, $\beta_1 = 0$; and $\mathbf{Q}^{n+1} := \tilde{\mathbf{Q}}^{\Upsilon}$ after final stage Υ

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Typical storage-efficient SSPRK(3,3):

$$\tilde{\mathbf{Q}}^1 = \mathbf{Q}^n + \Delta t \mathcal{F}(\mathbf{Q}^n), \quad \tilde{\mathbf{Q}}^2 = \frac{3}{4}\mathbf{Q}^n + \frac{1}{4}\tilde{\mathbf{Q}}^1 + \frac{1}{4}\Delta t \mathcal{F}(\tilde{\mathbf{Q}}^1),$$
$$\mathbf{Q}^{n+1} = \frac{1}{3}\mathbf{Q}^n + \frac{2}{3}\tilde{\mathbf{Q}}^2 + \frac{2}{3}\Delta t \mathcal{F}(\tilde{\mathbf{Q}}^2)$$

Outline

Elements of adaptive algorithms

- Base grid

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- Base grid
- Solver

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- Error indicators

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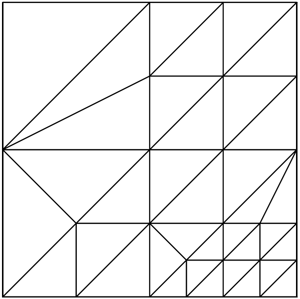
Adaptivity on unstructured meshes

- Coarse cells replaced by finer ones



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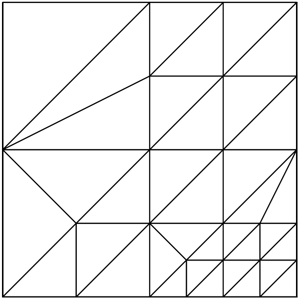

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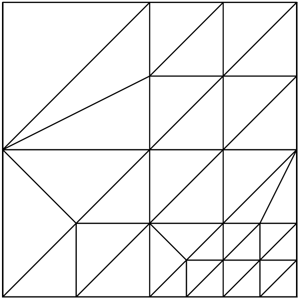
- Cell-based data structures



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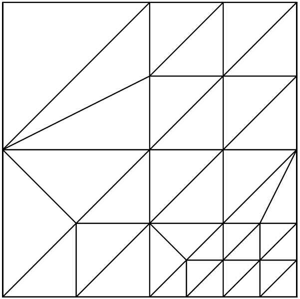

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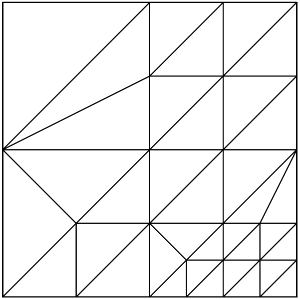

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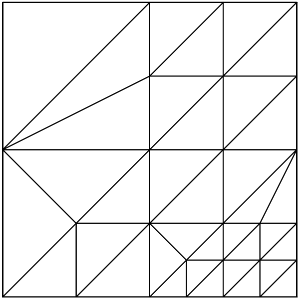

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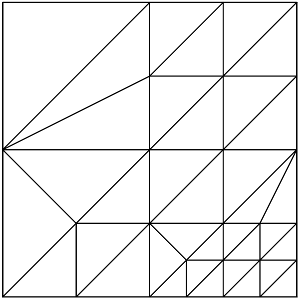

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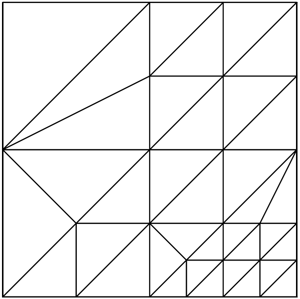

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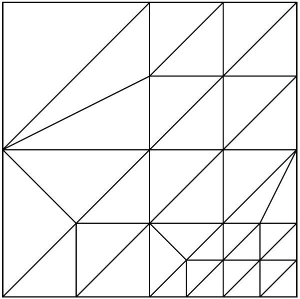

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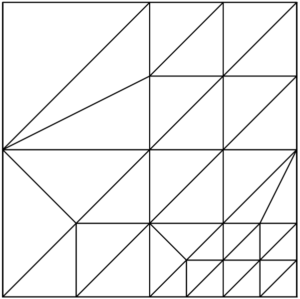

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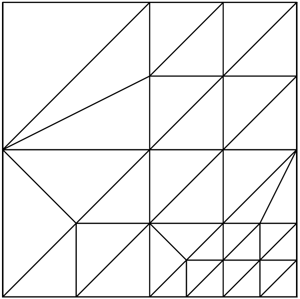

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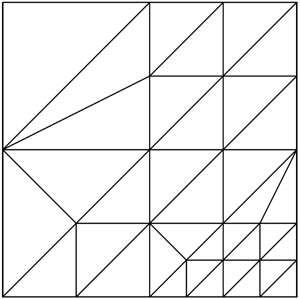

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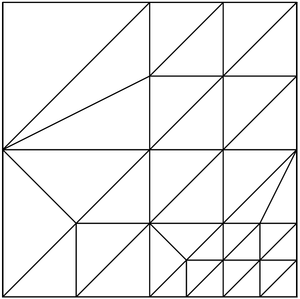

Structured mesh refinement techniques

- Block-based data of equal size

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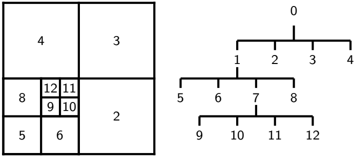

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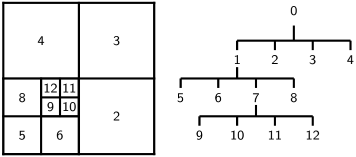

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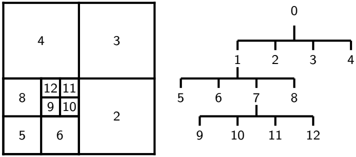

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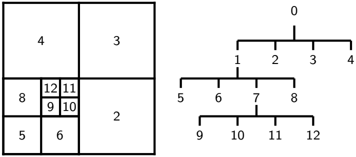

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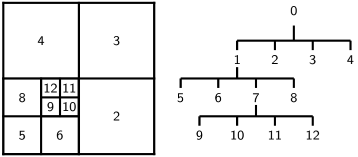

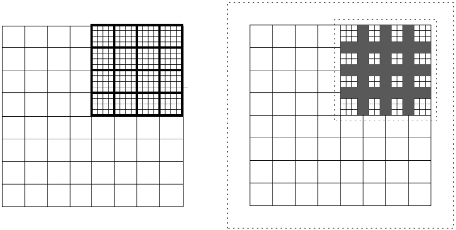

Wasted boundary space in a quad-tree

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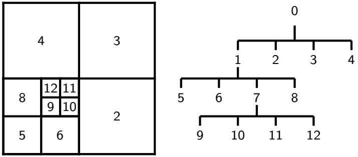

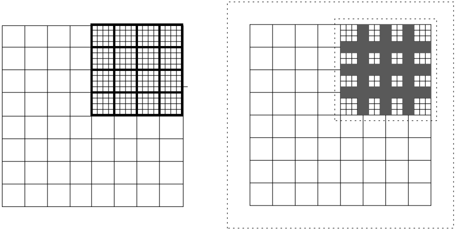

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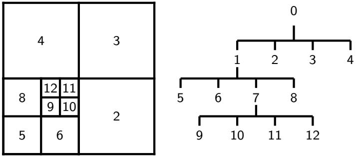

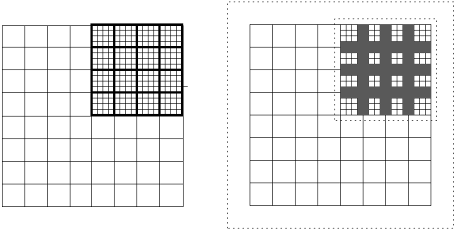

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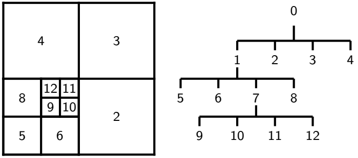

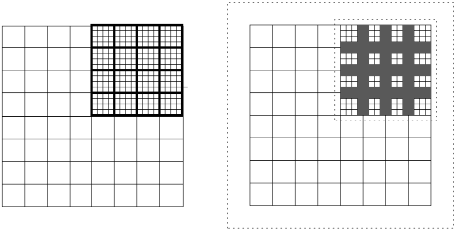

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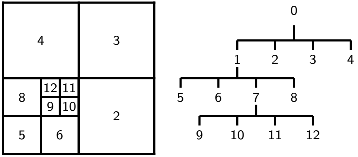

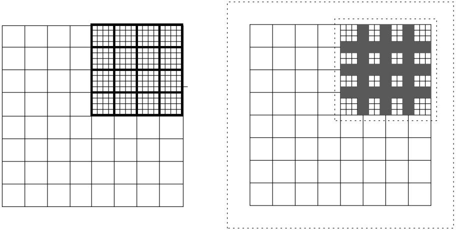

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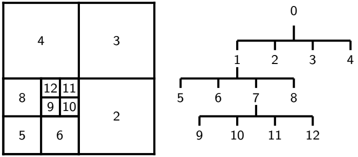
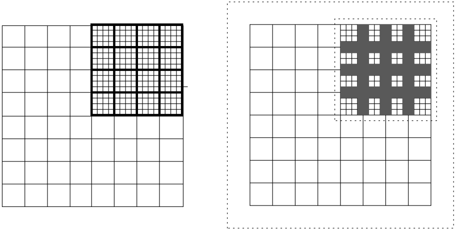
- \+ Parent/Child relations according to tree
- +/- Cache-reuse / vectorization only in data block

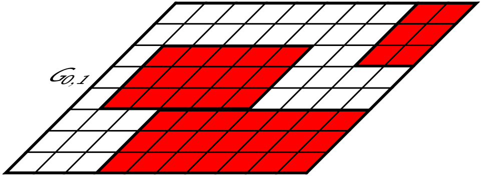




Wasted boundary space in a quad-tree

Block-structured adaptive mesh refinement (SAMR)

- Refined block overlay coarser ones



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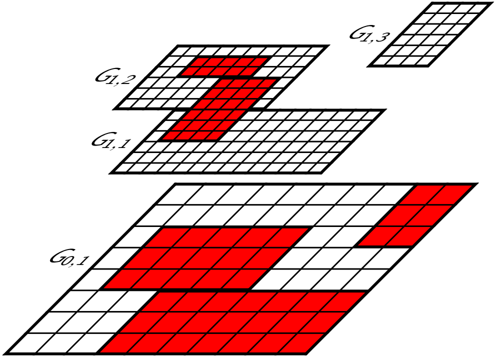

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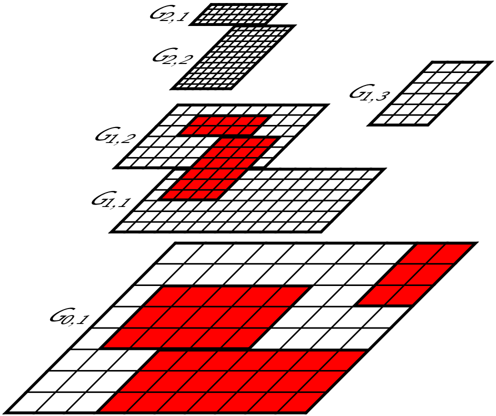

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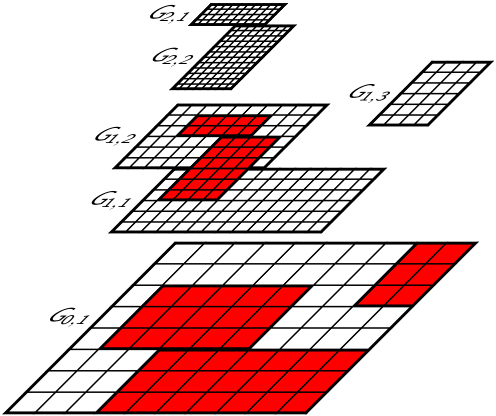

Block-structured adaptive mesh refinement (SAMR)

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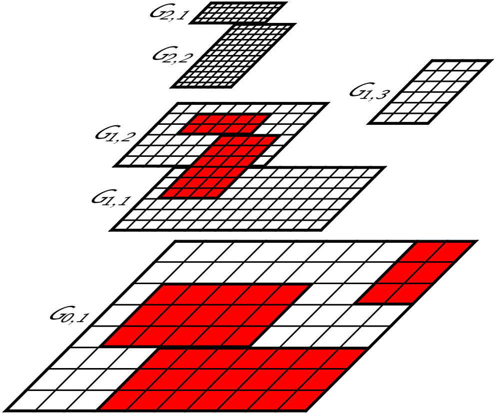
- Block (aka patch) based data structures



Block-structured adaptive mesh refinement (SAMR)

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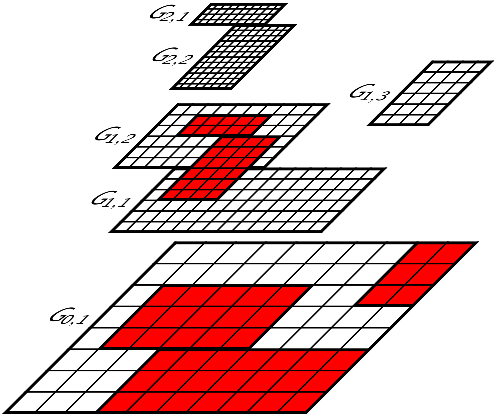

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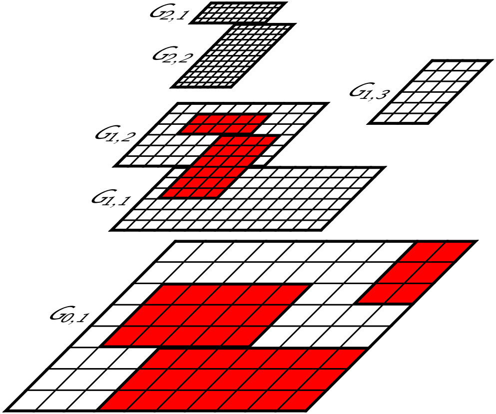

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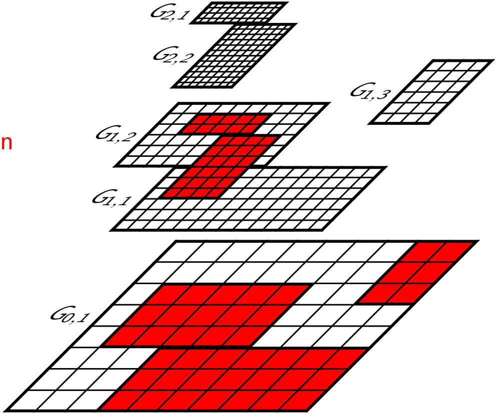
- \+ Numerical scheme only for single patch necessary
- \+ Efficient cache-reuse / vectorization possible



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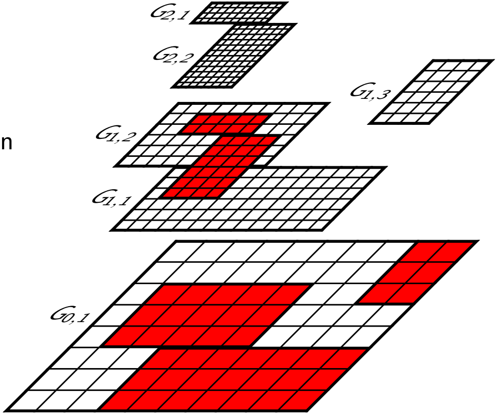

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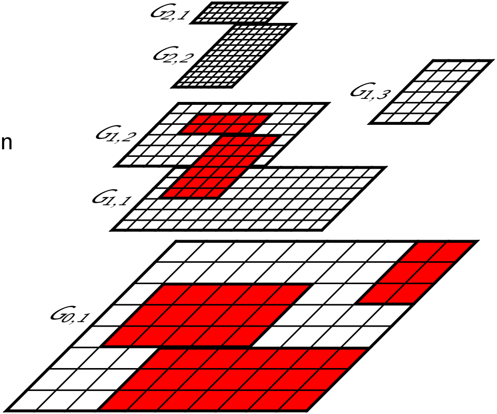
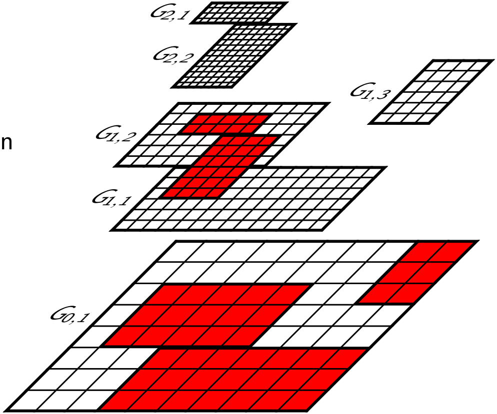


Block-structured adaptive mesh refinement (SAMR)

- Refined block overlay coarser ones
- Time-step refinement
- Block (aka patch) based data structures
- Global index coordinate system
- \+ Numerical scheme only for single patch necessary
- \+ Efficient cache-reuse / vectorization possible
- \+ Simple load-balancing
- \+ Minimal synchronization overhead
- \- Cells without mark are refined



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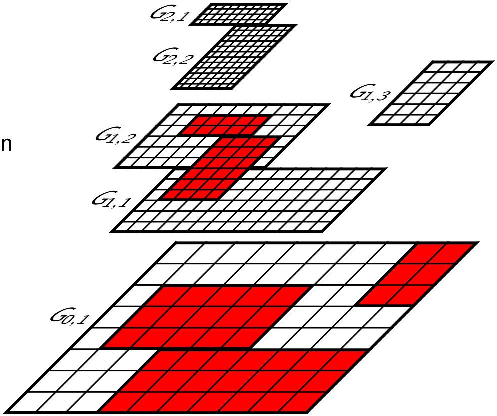

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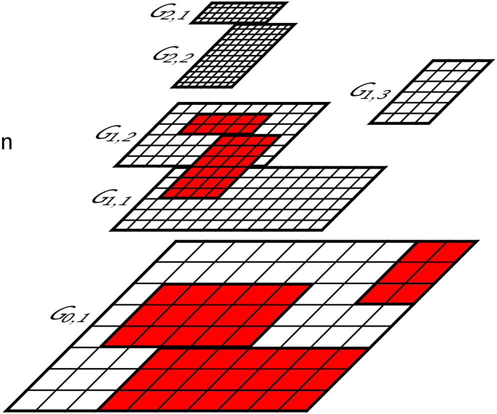

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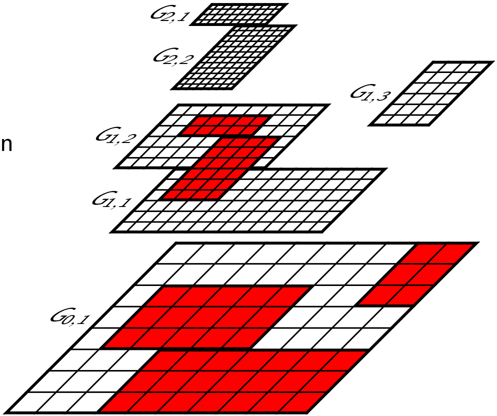
- \- Hanging nodes unavoidable
- \- Cluster-algorithm necessary
- \- Difficult to implement



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