Block-structured Adaptive Mesh Refinement Methods for Conservation Laws

Theory, Implementation and Application

Ralf Deiterding
Computer Science and Mathematics Division
Oak Ridge National Laboratory
P.O. Box 2008 MS6367, Oak Ridge, TN 37831, USA

E-mail: deiterdingr@ornl.gov

Structure of the lectures

1. Fundamentals
   - Finite volume schemes for hyperbolic problems
   - Discussion of mesh adaptation approaches
2. Structured AMR for hyperbolic problems
   - Presentation of all algorithmic components
   - Parallelization
3. Complex hyperbolic structured AMR applications
   - Shock-induced combustion
   - Fluid-structure interaction
4. Further topics
   - Using the SAMR approach for multigrid methods
   - Practical implementation, discussion of SAMR systems

Useful references I

Finite volume methods for hyperbolic problems

- LeVeque, R. J. (2002). *Finite volume methods for hyperbolic problems*. Cambridge University Press, Cambridge, New York.
- Godlewski, E. and Raviart, P.-A. (1996). *Numerical approximation of hyperbolic systems of conservation laws*. Springer Verlag, New York.
- Toro, E. F. (1999). *Riemann solvers and numerical methods for fluid dynamics*. Springer-Verlag, Berlin, Heidelberg, 2nd edition.
- Laney, C. B. (1998). *Computational gasdynamics*. Cambridge University Press, Cambridge.

Structured Adaptive Mesh Refinement

- Berger, M. and Colella, P. (1988). Local adaptive mesh refinement for shock hydrodynamics. *J. Comput. Phys.*, 82:64–84.
- Bell, J., Berger, M., Saltzman, J., and Welcome, M. (1994). Three-dimensional adaptive mesh refinement for hyperbolic conservation laws. *SIAM J. Sci. Comp.*, 15(1):127–138.
- Berger, M. and LeVeque, R. (1998). Adaptive mesh refinement using wave-propagation algorithms for hyperbolic systems. *SIAM J. Numer. Anal.*, 35(6):2298–2316.

Useful references II

Adaptive multigrid (finite difference and finite element based in textbooks)

- Hackbusch, W. (1985). *Multi-Grid Methods and Applications*. Springer Verlag, Berlin, Heidelberg.
- Briggs, W. L., Henson, V. E., and McCormick, S. F. (2001). *A Multigrid Tutorial*. Society for Industrial and Applied Mathematics, 2nd edition.
- Trottenberg, U., Oosterlee, C., and Schüller, A. (2001). *Multigrid*. Academic Press, San Antonio.
- Martin, D. F. (1998). *A cell-centered adaptive projection method for the incompressible Euler equations*. PhD thesis, University of California at Berkeley.

Implementation, parallelization

- Hornung, R. D., Wissink, A. M., and Kohn, S. H. (2006). Managing complex data and geometry in parallel structured AMR applications. *Engineering with Computers*, 22:181–195.
- Rendleman, C. A., Beckner, V. E., Lijewski, M., Crutchfield, W., and Bell, J. B. (2000). Parallelization of structured, hierarchical adaptive mesh refinement algorithms. *Computing and Visualization in Science*, 3:147–157.

Useful references III

- Deiterding, R. (2005). Construction and application of an AMR algorithm for distributed memory computers. In Plewa, T., Linde, T., and Weirs, V. G., editors, *Adaptive Mesh Refinement - Theory and Applications*, volume 41 of *Lecture Notes in Computational Science and Engineering*, pages 361–372. Springer.
- Deiterding, R. (2003). *Parallel adaptive simulation of multi-dimensional detonation structures*. PhD thesis, Brandenburgische Technische Universität Cottbus.

Applications (from my own work)

- Deiterding, R. (2009). A parallel adaptive method for simulating shock-induced combustion with detailed chemical kinetics in complex domains. *Computers & Structures*, 87:769–783.
- Deiterding, R., Radovitzky, R., Mauch, S. P., Noels, L., Cummings, J. C., and Meiron, D. I. (2006). A virtual test facility for the efficient simulation of solid materials under high energy shock-wave loading. *Engineering with Computers*, 22(3-4):325–347.
- Pantano, C., Deiterding, R., Hill, D. J., and Pullin, D. I. (2007). A low-numerical dissipation patch-based adaptive mesh refinement method for large-eddy simulation of compressible flows. *J. Comput. Phys.*, 221(1):63–87.

Lecture 1

Fundamentals: Used schemes and mesh adaptation

Course *Block-structured Adaptive Mesh Refinement Methods for Conservation Laws*
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Ralf Deiterding
Computer Science and Mathematics Division
Oak Ridge National Laboratory
P.O. Box 2008 MS6367, Oak Ridge, TN 37831, USA

E-mail: deiterdingr@ornl.gov

Outline

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Hyperbolic Conservation Laws

$$\frac{\partial}{\partial t}\mathbf{q}(\mathbf{x},t)+\sum_{n=1}^{d}\frac{\partial}{\partial x_n}\mathbf{f}_n(\mathbf{q}(\mathbf{x},t))=\mathbf{s}(\mathbf{q}(\mathbf{x},t)),\quad D\subset\{(\mathbf{x},t)\in\mathbb{R}^d\times\mathbb{R}_0^+\}$$

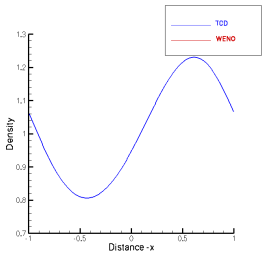
$\mathbf{q}=\mathbf{q}(\mathbf{x},t)\in S\subset\mathbb{R}^M$ - vector of state, $\mathbf{f}_n(\mathbf{q})\in \mathrm{C}^1(S,\mathbb{R}^M)$ - flux functions, $\mathbf{s}(\mathbf{q})\in \mathrm{C}^1(S,\mathbb{R}^M)$ - source term

Definition (Hyperbolicity)

$\mathbf{A}(\mathbf{q},\nu)=\nu_1\mathbf{A}_1(\mathbf{q})+\cdots+\nu_d\mathbf{A}_d(\mathbf{q})$ with $\mathbf{A}_n(\mathbf{q})=\partial\mathbf{f}_n(\mathbf{q})/\partial\mathbf{q}$ has M real eigenvalues $\lambda_1(\mathbf{q},\nu)\leq\ldots\leq\lambda_M(\mathbf{q},\nu)$ and M linear independent right eigenvectors $\mathbf{r}_m(\mathbf{q},\nu)$.

If $\mathbf{f}_n(\mathbf{q})$ is nonlinear, classical solutions $\mathbf{q}(\mathbf{x},t)\in \mathrm{C}^1(D,S)$ do not generally exist, not even for $\mathbf{q}_0(\mathbf{x})\in \mathrm{C}^1(\mathbb{R}^d,S)$ [Majda, 1984], [Godlewski and Raviart, 1996], [Kröner, 1997]

Example: Euler equations



Weak solutions

Integral form (Gauss's theorem):

$$\int_\Omega \mathbf{q}(\mathbf{x},t+\Delta t)\,d\mathbf{x}-\int_\Omega \mathbf{q}(\mathbf{x},t)\,d\mathbf{x}+\sum_{n=1}^{d}\int_t^{t+\Delta t}\int_{\partial\Omega}\mathbf{f}_n(\mathbf{q}(\mathbf{o},t))\,\sigma_n(\mathbf{o})\,d\mathbf{o}\,dt=\int_t^{t+\Delta t}\int_\Omega \mathbf{s}(\mathbf{q}(\mathbf{x},t))\,d\mathbf{x}$$

Theorem (Weak solution)

$\mathbf{q}_0\in \mathrm{L}^\infty_{loc}(\mathbb{R}^d,S)$. $\mathbf{q}\in \mathrm{L}^\infty_{loc}(D,S)$ *is weak solution if* $\mathbf{q}$ *satisfies*

$$\int_0^\infty\int_{\mathbb{R}^d}\left[\frac{\partial\varphi}{\partial t}\cdot\mathbf{q}+\sum_{n=1}^{d}\frac{\partial\varphi}{\partial x_n}\cdot\mathbf{f}_n(\mathbf{q})-\varphi\cdot\mathbf{s}(\mathbf{q})\right]d\mathbf{x}\,dt+\int_{\mathbb{R}^d}\varphi(\mathbf{x},0)\cdot\mathbf{q}_0(\mathbf{x})\,d\mathbf{x}=0$$

for any test function $\varphi\in \mathrm{C}^1_0(D,S)$

Entropy solutions

Select physical weak solution as $\lim_{\varepsilon\to 0}\mathbf{q}_\varepsilon=\mathbf{q}$ almost everywhere in D of

$$\frac{\partial\mathbf{q}_\varepsilon}{\partial t}+\sum_{n=1}^{d}\frac{\partial\mathbf{f}_n(\mathbf{q}_\varepsilon)}{\partial x_n}-\varepsilon\sum_{n=1}^{d}\frac{\partial^2\mathbf{q}_\varepsilon}{\partial x_n^2}=\mathbf{s}(\mathbf{q}_\varepsilon),\quad \mathbf{x}\in\mathbb{R}^d,\quad t>0$$

Theorem (Entropy condition)

Assume existence of entropy $\eta\in \mathrm{C}^2(S,\mathbb{R})$ *and entropy fluxes* $\psi_n\in \mathrm{C}^1(S,\mathbb{R})$ *that satisfy*

$$\frac{\partial\eta(\mathbf{q})}{\partial\mathbf{q}}^T\cdot\frac{\partial\mathbf{f}_n(\mathbf{q})}{\partial\mathbf{q}}=\frac{\partial\psi_n(\mathbf{q})}{\partial\mathbf{q}}^T,\quad n=1,\ldots,d$$

then $\lim_{\varepsilon\to 0}\mathbf{q}_\varepsilon=\mathbf{q}$ *almost everywhere in* D *is weak solution and satisfies*

$$\frac{\partial\eta(\mathbf{q})}{\partial t}+\sum_{n=1}^{d}\frac{\partial\psi_n(\mathbf{q})}{\partial x_n}\leq\frac{\partial\eta(\mathbf{q})}{\partial\mathbf{q}}^T\cdot\mathbf{s}(\mathbf{q})$$

in the sense of distributions. Proof: [Godlewski and Raviart, 1996]

Entropy solutions II

Definition (Entropy solution)

Weak solution $\mathbf{q}$ is called an entropy solution if $\mathbf{q}$ satisfies

$$\int_0^\infty\int_{\mathbb{R}^d}\left[\frac{\partial\varphi}{\partial t}\eta(\mathbf{q})+\sum_{n=1}^{d}\frac{\partial\varphi}{\partial x_n}\psi_n(\mathbf{q})-\varphi\frac{\partial\eta(\mathbf{q})}{\partial\mathbf{q}}^T\cdot\mathbf{s}(\mathbf{q})\right]d\mathbf{x}\,dt+\int_{\mathbb{R}^d}\varphi(\mathbf{x},0)\,\eta(\mathbf{q}_0(\mathbf{x}))\,d\mathbf{x}\geq 0$$

for all entropy functions $\eta(\mathbf{q})$ and all test functions $\varphi\in \mathrm{C}^1_0(D,\mathbb{R}^+_0)$, $\varphi\geq 0$

Theorem (Jump conditions)

An entropy solution $\mathbf{q}$ *is a classical solution* $\mathbf{q}\in \mathrm{C}^1(D,S)$ *almost everywhere and satisfies the Rankine-Hugoniot (RH) jump condition*

$$(\mathbf{q}^+-\mathbf{q}^-)\,\sigma_t+\sum_{n=1}^{d}(\mathbf{f}_n(\mathbf{q}^+)-\mathbf{f}_n(\mathbf{q}^-))\,\sigma_n=\mathbf{0}$$

and the jump inequality

$$(\eta(\mathbf{q}^+)-\eta(\mathbf{q}^-))\,\sigma_t+\sum_{n=1}^{d}(\psi_n(\mathbf{q}^+)-\psi_n(\mathbf{q}^-))\,\sigma_n\leq 0$$

along discontinuities. Proof: [Godlewski and Raviart, 1996]

Examples

Euler equations

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_n}(\rho u_n) = 0$$

$$\frac{\partial}{\partial t}(\rho u_k) + \frac{\partial}{\partial x_n}(\rho u_k u_n + \delta_{kn} p) = 0\,, \quad k = 1, \ldots, d$$

$$\frac{\partial}{\partial t}(\rho E) + \frac{\partial}{\partial x_n}(u_n(\rho E + p)) = 0$$

with polytrope gas equation of state

$$p = (\gamma - 1)(\rho E - \frac{1}{2}\rho u_n u_n)$$

have structure

$$\partial_t \mathbf{q}(\mathbf{x}, t) + \nabla \cdot \mathbf{f}(\mathbf{q}(\mathbf{x}, t)) = 0$$

Examples II

Navier-Stokes equations

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_n}(\rho u_n) = 0$$

$$\frac{\partial}{\partial t}(\rho u_k) + \frac{\partial}{\partial x_n}(\rho u_k u_n + \delta_{kn} p - \tau_{kn}) = 0\,, \quad k = 1, \ldots, d$$

$$\frac{\partial}{\partial t}(\rho E) + \frac{\partial}{\partial x_n}(u_n(\rho E + p) + q_n - \tau_{nj} u_j) = 0$$

with stress tensor

$$\tau_{kn} = \mu(\frac{\partial u_n}{\partial x_k} + \frac{\partial u_k}{\partial x_n}) - \frac{2}{3}\mu \frac{\partial u_j}{\partial x_j}\delta_{kn}$$

and heat conduction

$$q_n = -\lambda \frac{\partial T}{\partial x_n}$$

have structure

$$\partial_t \mathbf{q}(\mathbf{x}, t) + \nabla \cdot \mathbf{f}(\mathbf{q}(\mathbf{x}, t)) + \nabla \cdot \mathbf{h}(\mathbf{q}(\mathbf{x}, t), \nabla \mathbf{q}(\mathbf{x}, t)) = 0$$

Type can be either hyperbolic or parabolic

Outline

Derivation

Assume $\partial_t \mathbf{q} + \partial_x \mathbf{f}(\mathbf{q}) + \partial_x \mathbf{h}(\mathbf{q}(\cdot, \partial_x \mathbf{q})) = \mathbf{s}(\mathbf{q})$

Time discretization $t_n = n\Delta t$, discrete volumes
$I_j = [x_j - \frac{1}{2}\Delta x, x_j + \frac{1}{2}\Delta x[=: [x_{j-1/2}, x_{j+1/2}[$

Using approximations $\mathbf{Q}_j(t) \approx \frac{1}{|I_j|}\int_{I_j} \mathbf{q}(\mathbf{x}, t)\, dx, \quad \mathbf{s}(\mathbf{Q}_j(t)) \approx \frac{1}{|I_j|}\int_{I_j} \mathbf{s}(\mathbf{q}(\mathbf{x}, t))\, dx$

and numerical fluxes
$\mathbf{F}\left(\mathbf{Q}_j(t), \mathbf{Q}_{j+1}(t)\right) \approx \mathbf{f}(\mathbf{q}(x_{j+1/2}, t)), \quad \mathbf{H}\left(\mathbf{Q}_j(t), \mathbf{Q}_{j+1}(t)\right) \approx \mathbf{h}(\mathbf{q}(x_{j+1/2}, t), \nabla \mathbf{q}(x_{j+1/2}, t))$
yields after integration (Gauss theorem)

$$\mathbf{Q}_j(t_{n+1}) = \mathbf{Q}_j(t_n) - \frac{1}{\Delta x}\int_{t_n}^{t_{n+1}} [\mathbf{F}\left(\mathbf{Q}_j(t), \mathbf{Q}_{j+1}(t)\right) - \mathbf{F}\left(\mathbf{Q}_{j-1}(t), \mathbf{Q}_j(t)\right)]\, dt -$$

$$\frac{1}{\Delta x}\int_{t_n}^{t_{n+1}} [\mathbf{H}\left(\mathbf{Q}_j(t), \mathbf{Q}_{j+1}(t)\right) - \mathbf{H}\left(\mathbf{Q}_{j-1}(t), \mathbf{Q}_j(t)\right)]\, dt + \int_{t_n}^{t_{n+1}} \mathbf{s}(\mathbf{Q}_j(t))\, dt$$

For instance:

$$\mathbf{Q}_j^{n+1} = \mathbf{Q}_j^n - \frac{\Delta t}{\Delta x}\left[\mathbf{F}\left(\mathbf{Q}_j^n, \mathbf{Q}_{j+1}^n\right) - \mathbf{F}\left(\mathbf{Q}_{j-1}^n, \mathbf{Q}_j^n\right)\right] -$$

$$\frac{\Delta t}{\Delta x}\left[\mathbf{H}\left(\mathbf{Q}_j^n, \mathbf{Q}_{j+1}^n\right) - \mathbf{H}\left(\mathbf{Q}_{j-1}^n, \mathbf{Q}_j^n\right)\right] + \Delta t \mathbf{s}(\mathbf{Q}_j^n)\, dt$$

Some classical definitions

$(2s+1)$-point difference scheme of the form

$$\mathbf{Q}_j^{n+1} = \mathcal{H}^{(\Delta t)}(\mathbf{Q}_{j-s}^n, \ldots, \mathbf{Q}_{j+s}^n)$$

Definition (Stability)

For each time τ there is a constant C_S and a value $n_0 \in \mathbb{N}$ such that $\|\mathcal{H}^{(\Delta t)}(\mathbf{Q}^n)\| \leq C_S$ for all $n\Delta t \leq \tau$, $n < n_0$

Definition (Consistency)

If the local truncation error

$$\mathcal{L}^{(\Delta t)}(\mathbf{x}, t) := \frac{1}{\Delta t}\left[\mathbf{q}(\mathbf{x}, t+\Delta t) - \mathcal{H}^{(\Delta t)}(\mathbf{q}(\cdot, t))\right]$$

satisfies $\|\mathcal{L}^{(\Delta t)}(\cdot, t)\| \to 0$ as $\Delta t \to 0$

Definition (Convergence)

If the global error $\mathcal{E}^{(\Delta t)}(\mathbf{x}, t) := \mathbf{Q}(\mathbf{x}, t) - \mathbf{q}(\mathbf{x}, t)$ satisfies $\|\mathcal{E}^{(\Delta t)}(\cdot, t)\| \to 0$ as $\Delta t \to 0$ for all admissible initial data $\mathbf{q}_0(\mathbf{x})$

Some classical definitions II

Definition (Order of accuracy)

$\mathcal{H}(\cdot)$ is accurate of order o if for all sufficiently smooth initial data $\mathbf{q}_0(\mathbf{x})$, there is a constant C_L, such that the local truncation error satisfies $\|\mathcal{L}^{(\Delta t)}(\cdot, t)\| \leq C_L \Delta t^o$ for all $\Delta t < \Delta t_0$, $t \leq \tau$

Definition (Conservative form)

If $\mathcal{H}(\cdot)$ can be written in the form

$$\mathbf{Q}_j^{n+1} = \mathbf{Q}_j^n - \frac{\Delta t}{\Delta x}\left(\mathbf{F}(\mathbf{Q}_{j-s+1}^n, \ldots, \mathbf{Q}_{j+s}^n) - \mathbf{F}(\mathbf{Q}_{j-s}^n, \ldots, \mathbf{Q}_{j+s-1}^n)\right)$$

A conservative scheme satisfies

$$\sum_{j\in\mathbb{Z}} \mathbf{Q}_j^{n+1} = \sum_{j\in\mathbb{Z}} \mathbf{Q}_j^n$$

Definition (Consistency of a conservative method)

If the numerical flux satisfies $\mathbf{F}(\mathbf{q}, \ldots, \mathbf{q}) = \mathbf{f}(\mathbf{q})$ for all $\mathbf{q} \in S$

Splitting methods

Solve homogeneous PDE and ODE successively!

$$\mathcal{H}^{(\Delta t)}: \quad \partial_t \mathbf{q} + \nabla\cdot\mathbf{f}(\mathbf{q}) = 0\,, \quad \text{IC: } \mathbf{Q}(t_m) \overset{\Delta t}{\Longrightarrow} \tilde{\mathbf{Q}}$$
$$\mathcal{S}^{(\Delta t)}: \quad \partial_t \mathbf{q} = \mathbf{s}(\mathbf{q})\,, \quad \text{IC: } \tilde{\mathbf{Q}} \overset{\Delta t}{\Longrightarrow} \mathbf{Q}(t_m + \Delta t)$$

1st-order Godunov splitting: $\mathbf{Q}(t_m + \Delta t) = \mathcal{S}^{(\Delta t)}\mathcal{H}^{(\Delta t)}(\mathbf{Q}(t_m))$,
2nd-order Strang splitting : $\mathbf{Q}(t_m + \Delta t) = \mathcal{S}^{(\frac{1}{2}\Delta t)}\mathcal{H}^{(\Delta t)}\mathcal{S}^{(\frac{1}{2}\Delta t)}(\mathbf{Q}(t_m))$

1st-order dimensional splitting for $\mathcal{H}^{(\cdot)}$:

$$\mathcal{X}_1^{(\Delta t)}: \quad \partial_t \mathbf{q} + \partial_{x_1}\mathbf{f}_1(\mathbf{q}) = 0\,, \quad \text{IC: } \mathbf{Q}(t_m) \overset{\Delta t}{\Longrightarrow} \tilde{\mathbf{Q}}^{1/2}$$
$$\mathcal{X}_2^{(\Delta t)}: \quad \partial_t \mathbf{q} + \partial_{x_2}\mathbf{f}_2(\mathbf{q}) = 0\,, \quad \text{IC: } \tilde{\mathbf{Q}}^{1/2} \overset{\Delta t}{\Longrightarrow} \tilde{\mathbf{Q}}$$

[Toro, 1999]

Conservative scheme for diffusion equation

Consider $\partial_t q - c\Delta q = 0$ with $c \in \mathbb{R}^+$, which is readily discretized as

$$Q_{jk}^{n+1} = Q_{jk}^n + c\frac{\Delta t}{\Delta x_1^2}\left(Q_{j+1,k}^n - 2Q_{jk}^n + Q_{j-1,k}^n\right) + c\frac{\Delta t}{\Delta x_2^2}\left(Q_{j,k+1}^n - 2Q_{jk}^n + Q_{j,k-1}^n\right)$$

or conservatively

$$Q_{jk}^{n+1} = Q_{jk}^n + c\frac{\Delta t}{\Delta x_1}\left(H_{j+\frac{1}{2},k}^1 - H_{j-\frac{1}{2},k}^1\right) + c\frac{\Delta t}{\Delta x_2}\left(H_{j,k+\frac{1}{2}}^2 - H_{j,k-\frac{1}{2}}^2\right)$$

Von Neumann stability analysis: Insert single eigenmode $\hat{Q}(t)e^{ik_1x_1}e^{ik_2x_2}$ into discretization

$$\hat{Q}^{n+1} = \hat{Q}^n + C_1\left(\hat{Q}^n e^{ik_1\Delta x_1} - 2\hat{Q}^n + \hat{Q}^n e^{-ik_1\Delta x_1}\right) + C_2\left(\hat{Q}^n e^{ik_2\Delta x_2} - 2\hat{Q}^n + \hat{Q}^n e^{-ik_2\Delta x_2}\right)$$

with $C_\iota = c\frac{\Delta t}{\Delta x_\iota^2}$, $\iota = 1, 2$, which gives after inserting $e^{ik_\iota x_\iota} = \cos(k_\iota x_\iota) + i\sin(k_\iota x_\iota)$

$$\hat{Q}^{n+1} = \hat{Q}^n\left(1 + 2C_1(\cos(k_1\Delta x_1) - 1) + 2C_2(\cos(k_2\Delta x_2) - 1)\right)$$

Stability requires

$$|1 + 2C_1(\cos(k_1\Delta x_1) - 1) + 2C_2(\cos(k_2\Delta x_2) - 1)| \leq 1$$

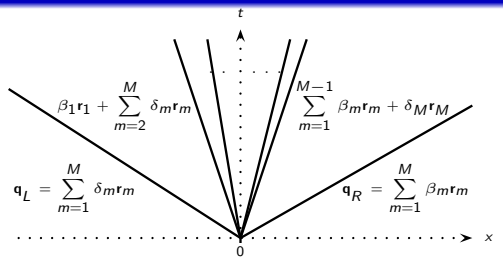
i.e.

$$|1 - 4C_1 - 4C_2| \leq 1$$

from which we derive the stability condition

$$0 \leq c\left(\frac{\Delta t}{\Delta x_1^2} + \frac{\Delta t}{\Delta x_2^2}\right) \leq \frac{1}{2}$$

Linear upwind schemes



Consider Riemann problem

$$\frac{\partial}{\partial t}\mathbf{q}(x,t)+\mathbf{A}\frac{\partial}{\partial x}\mathbf{q}(x,t)=\mathbf{0}\,,\quad x\in\mathbb{R}\,,\quad t>0$$

Has exact solution

$$\mathbf{q}(x,t)=\mathbf{q}_L+\sum_{\lambda_m<x/t} a_m\mathbf{r}_m=\mathbf{q}_R-\sum_{\lambda_m\geq x/t} a_m\mathbf{r}_m=\sum_{\lambda_m\geq x/t}\delta_m\mathbf{r}_m+\sum_{\lambda_m<x/t}\beta_m\mathbf{r}_m$$

Use Riemann problem to evaluate numerical flux $\mathbf{F}(\mathbf{q}_L,\mathbf{q}_R):=\mathbf{f}(\mathbf{q}(0,t))=\mathbf{A}\mathbf{q}(0,t)$ as

$$\mathbf{F}(\mathbf{q}_L,\mathbf{q}_R)=\mathbf{A}\mathbf{q}_L+\sum_{\lambda_m<0} a_m\lambda_m\mathbf{r}_m=\mathbf{A}\mathbf{q}_R-\sum_{\lambda_m\geq 0} a_m\lambda_m\mathbf{r}_m=\sum_{\lambda_m\geq 0}\delta_m\lambda_m\mathbf{r}_m+\sum_{\lambda_m<0}\beta_m\lambda_m\mathbf{r}_m$$

Use $\lambda_m^+=\max(\lambda_m,0)\,,\qquad \lambda_m^-=\min(\lambda_m,0)$

to define $\mathbf{\Lambda}^+:=\mathrm{diag}(\lambda_1^+,\ldots,\lambda_M^+)\,,\qquad \mathbf{\Lambda}^-:=\mathrm{diag}(\lambda_1^-,\ldots,\lambda_M^-)$

and $\mathbf{A}^+:=\mathbf{R}\,\mathbf{\Lambda}^+\,\mathbf{R}^{-1}\,,\qquad \mathbf{A}^-:=\mathbf{R}\,\mathbf{\Lambda}^-\,\mathbf{R}^{-1}$ which gives

$$\mathbf{F}(\mathbf{q}_L,\mathbf{q}_R)=\mathbf{A}\mathbf{q}_L+\mathbf{A}^-\Delta\mathbf{q}=\mathbf{A}\mathbf{q}_R-\mathbf{A}^+\Delta\mathbf{q}=\mathbf{A}^+\mathbf{q}_L+\mathbf{A}^-\mathbf{q}_R$$

with $\Delta\mathbf{q}=\mathbf{q}_R-\mathbf{q}_L$

Flux difference splitting

Godunov-type scheme with $\Delta\mathbf{Q}^n_{j+1/2}=\mathbf{Q}^n_{j+1}-\mathbf{Q}^n_j$

$$\mathbf{Q}^{n+1}_j=\mathbf{Q}^n_j-\frac{\Delta t}{\Delta x}\left(\mathbf{A}^-\Delta\mathbf{Q}^n_{j+1/2}+\mathbf{A}^+\Delta\mathbf{Q}^n_{j-1/2}\right)$$

Use linearization $\bar{\mathbf{f}}(\bar{\mathbf{q}})=\hat{\mathbf{A}}(\mathbf{q}_L,\mathbf{q}_R)\bar{\mathbf{q}}$ and construct scheme for nonlinear problem as

$$\mathbf{Q}^{n+1}_j=\mathbf{Q}^n_j-\frac{\Delta t}{\Delta x}\left(\hat{\mathbf{A}}^-(\mathbf{Q}^n_j,\mathbf{Q}^n_{j+1})\Delta\mathbf{Q}^n_{j+\frac{1}{2}}+\hat{\mathbf{A}}^+(\mathbf{Q}^n_{j-1},\mathbf{Q}^n_j)\Delta\mathbf{Q}^n_{j-\frac{1}{2}}\right)$$

stability condition

$$\max_{j\in\mathbb{Z}}|\hat{\lambda}_{m,j+\frac{1}{2}}|\frac{\Delta t}{\Delta x}\leq 1\,,\quad \text{for all } m=1,\ldots,M$$

[LeVeque, 1992]

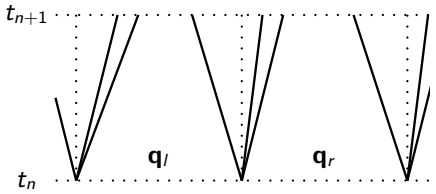
Roe's approximate Riemann solver

Choosing $\hat{\mathbf{A}}(\mathbf{q}_L,\mathbf{q}_R)$ [Roe, 1981]:

(i) $\hat{\mathbf{A}}(\mathbf{q}_L,\mathbf{q}_R)$ has real eigenvalues

(ii) $\hat{\mathbf{A}}(\mathbf{q}_L,\mathbf{q}_R)\to\dfrac{\partial\mathbf{f}(\mathbf{q})}{\partial\mathbf{q}}$ as $\mathbf{q}_L,\mathbf{q}_R\to\mathbf{q}$

(iii) $\hat{\mathbf{A}}(\mathbf{q}_L,\mathbf{q}_R)\Delta\mathbf{q}=\mathbf{f}(\mathbf{q}_R)-\mathbf{f}(\mathbf{q}_L)$



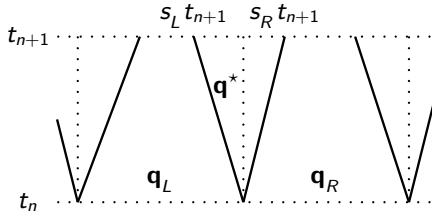
For Euler equations:

$$\hat{\rho}=\frac{\sqrt{\rho_L}\rho_R+\sqrt{\rho_R}\rho_L}{\sqrt{\rho_L}+\sqrt{\rho_R}}=\sqrt{\rho_L\rho_R}\quad\text{and}\quad\hat{v}=\frac{\sqrt{\rho_L}v_L+\sqrt{\rho_R}v_R}{\sqrt{\rho_L}+\sqrt{\rho_R}}\quad\text{for } v=u_n,H$$

Wave decomposition: $\Delta\mathbf{q}=\mathbf{q}_r-\mathbf{q}_l=\sum_m a_m\,\hat{\mathbf{r}}_m$

$$\begin{aligned}\mathbf{F}(\mathbf{q}_L,\mathbf{q}_R)&=\mathbf{f}(\mathbf{q}_L)+\sum_{\hat{\lambda}_m<0}\hat{\lambda}_m\,a_m\,\hat{\mathbf{r}}_m=\mathbf{f}(\mathbf{q}_R)-\sum_{\hat{\lambda}_m\geq 0}\hat{\lambda}_m\,a_m\,\hat{\mathbf{r}}_m\\&=\frac{1}{2}\left(\mathbf{f}(\mathbf{q}_L)+\mathbf{f}(\mathbf{q}_R)-\sum_m|\hat{\lambda}_m|\,a_m\,\hat{\mathbf{r}}_m\right)\end{aligned}$$

Harten-Lax-Van Leer (HLL) approximate Riemann solver



$$\bar{\mathbf{q}}(x,t)=\begin{cases}\mathbf{q}_L\,, & x<s_L\,t\\ \mathbf{q}^\star\,, & s_L\,t\leq x\leq s_R\,t\\ \mathbf{q}_R\,, & x>s_R\,t\end{cases}$$

$$\mathbf{F}_{HLL}(\mathbf{q}_L,\mathbf{q}_R)=\begin{cases}\mathbf{f}(\mathbf{q}_L)\,, & 0<s_L\,,\\ \dfrac{s_R\mathbf{f}(\mathbf{q}_L)-s_L\mathbf{f}(\mathbf{q}_R)+s_Ls_R(\mathbf{q}_R-\mathbf{q}_L)}{s_R-s_L}\,, & s_L\leq 0\leq s_R\,,\\ \mathbf{f}(\mathbf{q}_R)\,, & 0>s_R\,,\end{cases}$$

Euler equations:

$$s_L=\min(u_{1,L}-c_L,u_{1,R}-c_R)\,,\quad s_R=\max(u_{1,L}+c_l,u_{1,R}+c_R)$$

[Toro, 1999], HLLC: [Toro et al., 1994]

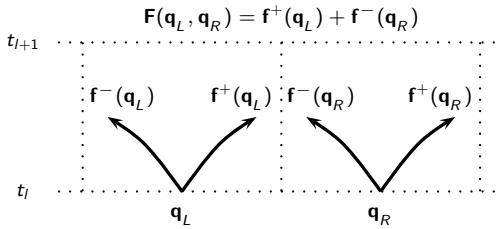
Flux vector splitting

Splitting

$$\mathbf{f}(\mathbf{q}) = \mathbf{f}^+(\mathbf{q}) + \mathbf{f}^-(\mathbf{q})$$

derived under restriction $\hat{\lambda}_m^+ \geq 0$ and $\hat{\lambda}_m^- \leq 0$ for all $m = 1, \ldots, M$ for

$$\hat{\mathbf{A}}^+(\mathbf{q}) = \frac{\partial \mathbf{f}^+(\mathbf{q})}{\partial \mathbf{q}}, \quad \hat{\mathbf{A}}^-(\mathbf{q}) = \frac{\partial \mathbf{f}^-(\mathbf{q})}{\partial \mathbf{q}}$$



plus reproduction of regular upwinding

$$\begin{array}{llllllll} \mathbf{f}^+(\mathbf{q}) & = & \mathbf{f}(\mathbf{q}), & \mathbf{f}^-(\mathbf{q}) & = & \mathbf{0} & \text{if} \quad \lambda_m \geq 0 \quad \text{for all} \quad m = 1, \ldots, M \\ \mathbf{f}^+(\mathbf{q}) & = & \mathbf{0}, & \mathbf{f}^-(\mathbf{q}) & = & \mathbf{f}(\mathbf{q}) & \text{if} \quad \lambda_m \leq 0 \quad \text{for all} \quad m = 1, \ldots, M \end{array}$$

Then use

$$\mathbf{F}(\mathbf{q}_L, \mathbf{q}_R) = \mathbf{f}^+(\mathbf{q}_L) + \mathbf{f}^-(\mathbf{q}_R)$$

Steger-Warming

Required $\mathbf{f}(\mathbf{q}) = \mathbf{A}(\mathbf{q})\,\mathbf{q}$

$$\lambda_m^+ = \frac{1}{2}\left(\lambda_m + |\lambda_m|\right) \qquad \lambda_m^- = \frac{1}{2}\left(\lambda_m - |\lambda_m|\right)$$

$$\mathbf{A}^+(\mathbf{q}) := \mathbf{R}(\mathbf{q})\,\mathbf{\Lambda}^+(\mathbf{q})\,\mathbf{R}^{-1}(\mathbf{q}), \qquad \mathbf{A}^-(\mathbf{q}) := \mathbf{R}(\mathbf{q})\,\mathbf{\Lambda}^-(\mathbf{q})\,\mathbf{R}^{-1}(\mathbf{q})$$

Gives

$$\mathbf{f}(\mathbf{q}) = \mathbf{A}^+(\mathbf{q})\,\mathbf{q} + \mathbf{A}^-(\mathbf{q})\,\mathbf{q}$$

and the numerical flux

$$\mathbf{F}(\mathbf{q}_L, \mathbf{q}_R) = \mathbf{A}^+(\mathbf{q}_L)\,\mathbf{q}_L + \mathbf{A}^-(\mathbf{q}_R)\,\mathbf{q}_R$$

Jacobians of the split fluxes are identical to $\mathbf{A}^\pm(\mathbf{q})$ only in linear case

$$\frac{\partial \mathbf{f}^\pm(\mathbf{q})}{\partial \mathbf{q}} = \frac{\partial\left(\mathbf{A}^\pm(\mathbf{q})\,\mathbf{q}\right)}{\partial \mathbf{q}} = \mathbf{A}^\pm(\mathbf{q}) + \frac{\partial \mathbf{A}^\pm(\mathbf{q})}{\partial \mathbf{q}}\,\mathbf{q}$$

Further methods: Van Leer FVS [Toro, 1999], AUSM [Wada and Liou, 1997]

High-resolution methods

Objective: Higher-order accuracy in smooth solution regions but no spurious oscillations near large gradients
Consistent monotone methods converge toward the entropy solution, but

Theorem
A monotone method is at most first order accurate.

Proof: [Harten et al., 1976]

Definition (TVD property)
Scheme $\mathcal{H}^{(\Delta t)}(\mathbf{Q}^n; j)$ TVD if $TV(\mathbf{Q}^{l+1}) \leq TV(\mathbf{Q}^l)$ is satisfied for all discrete sequences $\mathbf{Q}^n$. Herein, $TV(\mathbf{Q}^l) := \sum_{j\in\mathbb{Z}} |\mathbf{Q}_{j+1}^l - \mathbf{Q}_j^l|$.

TVD schemes: no new extrema, local minima are non-decreasing, local maxima are non-increasing (termed *monotonicity-preserving*). *Monotonicity-preserving* higher-order schemes are at least 5-point methods. Proofs: [Harten, 1983]

TVD concept is proven [Godlewski and Raviart, 1996] for scalar schemes only but nevertheless used to construct *high resolution* schemes.
Monotonicity-preserving scheme can converge toward non-physical weak solutions.

MUSCL slope limiting

Monotone Upwind Schemes for Conservation Laws [van Leer, 1979]

$$\tilde{Q}_{j+\frac{1}{2}}^L = Q_j^n + \frac{1}{4}\left[(1-\omega)\,\Phi_{j-\frac{1}{2}}^+\,\Delta_{j-\frac{1}{2}} + (1+\omega)\,\Phi_{j+\frac{1}{2}}^-\,\Delta_{j+\frac{1}{2}}\right],$$

$$\tilde{Q}_{j-\frac{1}{2}}^R = Q_j^n - \frac{1}{4}\left[(1-\omega)\,\Phi_{j+\frac{1}{2}}^-\,\Delta_{j+\frac{1}{2}} + (1+\omega)\,\Phi_{j-\frac{1}{2}}^+\,\Delta_{j-\frac{1}{2}}\right]$$

with $\Delta_{j-1/2} = Q_j^n - Q_{j-1}^n$, $\Delta_{j+1/2} = Q_{j+1}^n - Q_j^n$.

$$\Phi_{j-\frac{1}{2}}^+ := \Phi\left(r_{j-\frac{1}{2}}^+\right), \quad \Phi_{j+\frac{1}{2}}^- := \Phi\left(r_{j+\frac{1}{2}}^-\right) \quad \text{with} \quad r_{j-\frac{1}{2}}^+ := \frac{\Delta_{j+\frac{1}{2}}}{\Delta_{j-\frac{1}{2}}}, \quad r_{j+\frac{1}{2}}^- := \frac{\Delta_{j-\frac{1}{2}}}{\Delta_{j+\frac{1}{2}}}$$

and *slope limiters*, e.g., *Minmod*

$$\Phi(r) = \max(0, \min(r, 1))$$

Using a midpoint rule for temporal integration, e.g.,

$$Q_j^\star = Q_j^n - \frac{1}{2}\frac{\Delta t}{\Delta x}\left(F(Q_{j+1}^n, Q_j^n) - F(Q_j^n, Q_{j-1}^n)\right)$$

and constructing limited values from $Q^\star$ to be used in FV scheme gives a TVD method if

$$\frac{1}{2}\left[(1-\omega)\Phi(r) + (1+\omega)\,r\,\Phi\left(\frac{1}{r}\right)\right] < \min(2, 2r)$$

is satisfied for $r > 0$. Proof: [Hirsch, 1988]

Wave Propagation with flux limiting

Wave Propagation Method [LeVeque, 1997] is built on the flux differencing approach $\mathcal{A}^{\pm}\Delta := \hat{\mathbf{A}}^{\pm}(\mathbf{q}_L, \mathbf{q}_R)\Delta\mathbf{q}$ and the *waves* $\mathcal{W}_m := a_m \hat{\mathbf{r}}_m$, i.e.

$$\mathcal{A}^{-}\Delta\mathbf{q} = \sum_{\hat{\lambda}_m<0} \hat{\lambda}_m \mathcal{W}_m\,, \quad \mathcal{A}^{+}\Delta\mathbf{q} = \sum_{\hat{\lambda}_m\geq 0} \hat{\lambda}_m \mathcal{W}_m$$

Wave Propagation 1D:

$$\mathbf{Q}^{n+1} = \mathbf{Q}_j^n - \frac{\Delta t}{\Delta x}\left(\mathcal{A}^{-}\Delta_{j+\frac{1}{2}} + \mathcal{A}^{+}\Delta_{j-\frac{1}{2}}\right) - \frac{\Delta t}{\Delta x}\left(\tilde{\mathbf{F}}_{j+\frac{1}{2}} - \tilde{\mathbf{F}}_{j-\frac{1}{2}}\right)$$

with

$$\tilde{\mathbf{F}}_{j+\frac{1}{2}} = \frac{1}{2}|\mathcal{A}|\left(1 - \frac{\Delta t}{\Delta x}|\mathcal{A}|\right)\Delta_{j+\frac{1}{2}} = \frac{1}{2}\sum_{m=1}^{M}|\hat{\lambda}^m_{j+\frac{1}{2}}|\left(1 - \frac{\Delta t}{\Delta x}\right)|\hat{\lambda}^m_{j+\frac{1}{2}}|\,\tilde{\mathcal{W}}^m_{j+\frac{1}{2}}$$

and wave limiter

$$\tilde{\mathcal{W}}^m_{j+\frac{1}{2}} = \Phi(\Theta^m_{j+\frac{1}{2}})\,\mathcal{W}^m_{j+\frac{1}{2}}$$

with

$$\Theta^m_{j+\frac{1}{2}} = \begin{cases} a^m_{j-\frac{1}{2}}/a^m_{j+\frac{1}{2}}\,, & \hat{\lambda}^m_{j+\frac{1}{2}} \geq 0\,, \\ a^m_{j+\frac{3}{2}}/a^m_{j+\frac{1}{2}}\,, & \hat{\lambda}^m_{j+\frac{1}{2}} < 0 \end{cases}$$

Wave Propagation Method in 2D

Writing $\tilde{\mathcal{A}}^{\pm}\Delta_{j\pm1/2} := \mathcal{A}^{+}\Delta_{j\pm1/2} + \tilde{\mathbf{F}}_{j\pm1/2}$ one can develop a truly two-dimensional one-step method [Langseth and LeVeque, 2000]

$$\begin{aligned}\mathbf{Q}^{n+1}_{jk} = \mathbf{Q}^n_{jk} &- \frac{\Delta t}{\Delta x_1}\Big(\tilde{\mathcal{A}}^{-}\Delta_{j+\frac{1}{2},k} - \frac{1}{2}\frac{\Delta t}{\Delta x_2}\left[\mathcal{A}^{-}\tilde{\mathcal{B}}^{-}\Delta_{j+1,k+\frac{1}{2}} + \mathcal{A}^{-}\tilde{\mathcal{B}}^{+}\Delta_{j+1,k-\frac{1}{2}}\right] + \\ &\tilde{\mathcal{A}}^{+}\Delta_{j-\frac{1}{2},k} - \frac{1}{2}\frac{\Delta t}{\Delta x_2}\left[\mathcal{A}^{+}\tilde{\mathcal{B}}^{-}\Delta_{j-1,k+\frac{1}{2}} + \mathcal{A}^{+}\tilde{\mathcal{B}}^{+}\Delta_{j-1,k-\frac{1}{2}}\right]\Big) \\ &- \frac{\Delta t}{\Delta x_2}\Big(\tilde{\mathcal{B}}^{-}\Delta_{j,k+\frac{1}{2}} - \frac{1}{2}\frac{\Delta t}{\Delta x_1}\left[\mathcal{B}^{-}\tilde{\mathcal{A}}^{-}\Delta_{j+\frac{1}{2},k+1} + \mathcal{B}^{-}\tilde{\mathcal{A}}^{+}\Delta_{j-\frac{1}{2},k+1}\right] + \\ &\tilde{\mathcal{B}}^{+}\Delta_{j,k-\frac{1}{2}} - \frac{1}{2}\frac{\Delta t}{\Delta x_1}\left[\mathcal{B}^{+}\tilde{\mathcal{A}}^{-}\Delta_{j+\frac{1}{2},k-1} + \mathcal{B}^{+}\tilde{\mathcal{A}}^{+}\Delta_{j-\frac{1}{2},k-1}\right]\Big)\end{aligned}$$

that is stable for

$$\left\{\max_{j\in\mathbb{Z}}|\hat{\lambda}_{m,j+\frac{1}{2}}|\frac{\Delta t}{\Delta x_1}, \max_{k\in\mathbb{Z}}|\hat{\lambda}_{m,k+\frac{1}{2}}|\frac{\Delta t}{\Delta x_2}\right\} \leq 1\,, \quad \text{for all } m = 1,\ldots,M$$

Further high-resolution methods

Some further high-resolution methods (good overview in [Laney, 1998]):

- FCT: 2nd order [Oran and Boris, 2001]
- ENO/WENO: 3rd order [Shu, 97]
- PPM: 3rd order [Colella and Woodward, 1984]

3rd order methods must make use of strong-stability preserving Runge-Kutta methods [Gottlieb et al., 2001] for time integration that use a multi-step update

$$\tilde{\mathbf{Q}}^{\upsilon}_j = \alpha_\upsilon \mathbf{Q}^n_j + \beta_\upsilon \tilde{\mathbf{Q}}^{\upsilon-1}_j + \gamma_\upsilon \frac{\Delta t}{\Delta x}\left(\mathbf{F}_{j+\frac{1}{2}}(\tilde{\mathbf{Q}}^{\upsilon-1}) - \mathbf{F}_{j-\frac{1}{2}}(\tilde{\mathbf{Q}}^{\upsilon-1})\right)$$

with $\tilde{\mathbf{Q}}^0 := \mathbf{Q}^n$, $\alpha_1 = 1$, $\beta_1 = 0$; and $\mathbf{Q}^{n+1} := \tilde{\mathbf{Q}}^{\Upsilon}$ after final stage Υ

Typical storage-efficient SSPRK(3,3):

$$\tilde{\mathbf{Q}}^1 = \mathbf{Q}^n + \Delta t\mathcal{F}(\mathbf{Q}^n), \quad \tilde{\mathbf{Q}}^2 = \frac{3}{4}\mathbf{Q}^n + \frac{1}{4}\tilde{\mathbf{Q}}^1 + \frac{1}{4}\Delta t\mathcal{F}(\tilde{\mathbf{Q}}^1),$$
$$\mathbf{Q}^{n+1} = \frac{1}{3}\mathbf{Q}^n + \frac{2}{3}\tilde{\mathbf{Q}}^2 + \frac{2}{3}\Delta t\mathcal{F}(\tilde{\mathbf{Q}}^2)$$

Outline

Elements of adaptive algorithms

- Base grid
- Solver
- Error indicators
- Grid manipulation
- Interpolation (restriction and prolongation)
- Load-balancing

Adaptivity on unstructured meshes

- Coarse cells replaced by finer ones
- Global time-step
- Cell-based data structures
- Neighborhoods have to stored

\+ Geometric flexible

\+ No hanging nodes

\+ Easy to implement

\- Higher order difficult to achieve

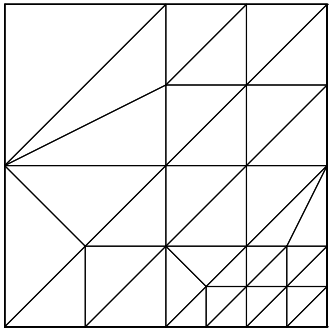
\- Cell aspect ratio must be considered

\- Fragmented data

\- Cache-reuse / vectorizaton nearly impossible

\- Complex load-balancing

\- Complex synchronization



Structured mesh refinement techniques

- Block-based data of equal size
- Block stored in a quad-tree
- Time-step refinement
- Global index coordinate system
- Neighborhoods need not be stored

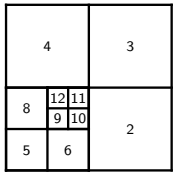
\+ Numerical scheme only for single regular block necessary

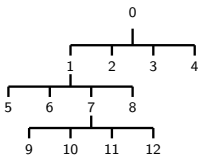
\+ Easy to implement

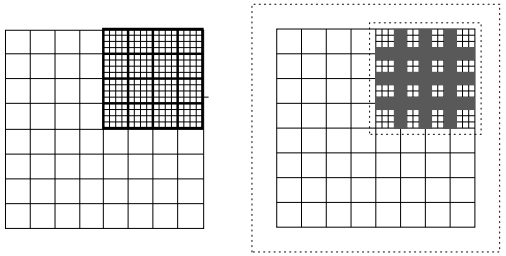
\+ Simple load-balancing

\+ Parent/Child relations according to tree

+/- Cache-reuse / vectorization only in data block







Wasted boundary space in a quad-tree

Block-structured adaptive mesh refinement (SAMR)

- Refined block overlay coarser ones
- Time-step refinement
- Block (aka patch) based data structures
- Global index coordinate system

\+ Numerical scheme only for single patch necessary

\+ Efficient cache-reuse / vectorization possible

\+ Simple load-balancing

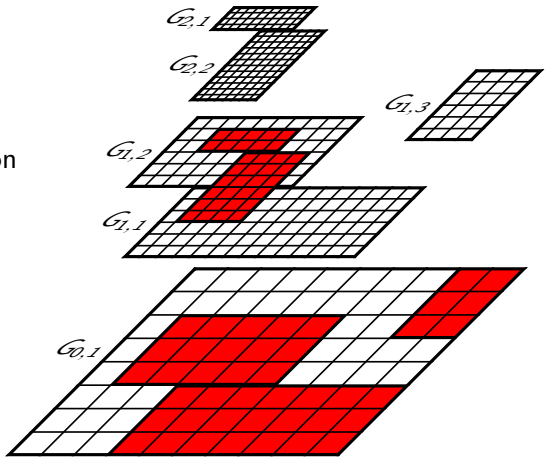
\+ Minimal synchronization overhead

\- Cells without mark are refined

\- Hanging nodes unavoidable

\- Cluster-algorithm necessary

\- Difficult to implement



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Lecture 2

The SAMR method for hyperbolic problems

Course *Block-structured Adaptive Mesh Refinement Methods for Conservation Laws*
Theory, Implementation and Application

Ralf Deiterding
Computer Science and Mathematics Division
Oak Ridge National Laboratory
P.O. Box 2008 MS6367, Oak Ridge, TN 37831, USA

E-mail: deiterdingr@ornl.gov

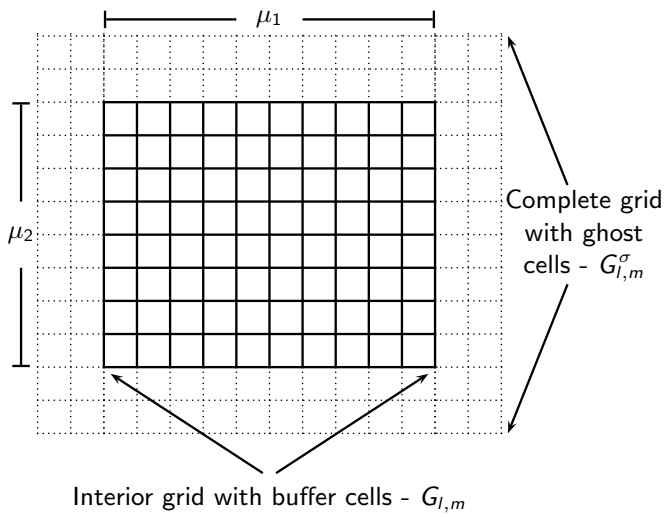
Outline

Outline

The mth refinement grid on level l

Notations:

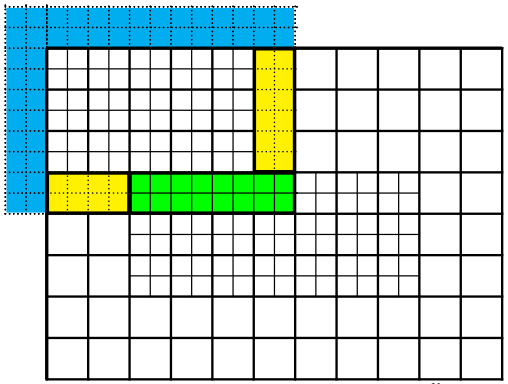
- Boundary: $\partial G_{l,m}$
- Hull: $\bar{G}_{l,m} = G_{l,m} \cup \partial G_{l,m}$
- Ghost cell region: $\tilde{G}^{\sigma}_{l,m} = G^{\sigma}_{l,m} \backslash \bar{G}_{l,m}$



Refinement data

- Resolution: $\Delta t_l := \dfrac{\Delta t_{l-1}}{r_l}$ and $\Delta x_{n,l} := \dfrac{\Delta x_{n,l-1}}{r_l}$
- Refinement factor: $r_l \in \mathbb{N}, r_l \geq 2$ for $l > 0$ and $r_0 = 1$
- Integer coordinate system for internal organization [Bell et al., 1994]:
  $\Delta x_{n,l} \cong \prod_{\kappa=l+1}^{l_{\max}} r_\kappa$
- Computational Domain: $G_0 = \bigcup_{m=1}^{M_0} G_{0,m}$
- Domain of level l: $G_l := \bigcup_{m=1}^{M_l} G_{l,m}$ with $G_{l,m} \cap G_{l,n} = \emptyset$ for $m \neq n$
- Refinements are properly nested: $G^1_l \subset G_{l-1}$
- Assume a FD scheme with stencil radius s. Necessary data:
  - Vector of state: $\mathbf{Q}^l := \bigcup_m \mathbf{Q}(G^s_{l,m})$
  - Numerical fluxes: $\mathbf{F}^{n,l} := \bigcup_m \mathbf{F}^n(\bar{G}_{l,m})$
  - Flux corrections: $\delta\mathbf{F}^{n,l} := \bigcup_m \delta\mathbf{F}^n(\partial G_{l,m})$

Setting of ghost cells



Synchronization with G_l - $\tilde{S}^s_{l,m} = \tilde{G}^s_{l,m} \cap G_l$
Physical boundary conditions - $\tilde{P}^s_{l,m} = \tilde{G}^s_{l,m} \backslash G_0$
Interpolation from G_{l-1} - $\tilde{I}^s_{l,m} = \tilde{G}^s_{l,m} \backslash (\tilde{S}^s_{l,m} \cup \tilde{P}^s_{l,m})$

Numerical update

Time-explicit conservative finite volume scheme

$$\mathcal{H}^{(\Delta t)}: \ \mathbf{Q}_{jk}(t+\Delta t) = \mathbf{Q}_{jk}(t) - \frac{\Delta t}{\Delta x_1}\left(\mathbf{F}^1_{j+\frac{1}{2},k} - \mathbf{F}^1_{j-\frac{1}{2},k}\right) - \frac{\Delta t}{\Delta x_2}\left(\mathbf{F}^2_{j,k+\frac{1}{2}} - \mathbf{F}^2_{j,k-\frac{1}{2}}\right)$$

`UpdateLevel(`l`)`

`For all` $m = 1$ `To` M_l `Do`
  $\mathbf{Q}(G^s_{l,m}, t) \xrightarrow{\mathcal{H}^{(\Delta t_l)}} \mathbf{Q}(G_{l,m}, t + \Delta t_l)\ , \mathbf{F}^n(\bar{G}_{l,m}, t)$
  `If level` $l > 0$
    `Add` $\mathbf{F}^n(\partial G_{l,m}, t)$ `to` $\delta\mathbf{F}^{n,l}$
  `If level` $l+1$ `exists`
    `Init` $\delta\mathbf{F}^{n,l+1}$ `with` $\mathbf{F}^n(\bar{G}_{l,m} \cap \partial G_{l+1}, t)$

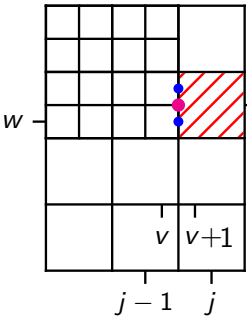
Conservative flux correction

Example: Cell j, k

$$\check{\mathbf{Q}}^l_{jk}(t+\Delta t_l) = \mathbf{Q}^l_{jk}(t) - \frac{\Delta t_l}{\Delta x_{1,l}}\left(\mathbf{F}^{1,l}_{j+\frac{1}{2},k} - \frac{1}{r^2_{l+1}}\sum_{\kappa=0}^{r_{l+1}-1}\sum_{\iota=0}^{r_{l+1}-1}\mathbf{F}^{1,l+1}_{v+\frac{1}{2},w+\iota}(t+\kappa\Delta t_{l+1})\right) - \frac{\Delta t_l}{\Delta x_{2,l}}\left(\mathbf{F}^{2,l}_{j,k+\frac{1}{2}} - \mathbf{F}^{2,l}_{j,k-\frac{1}{2}}\right)$$

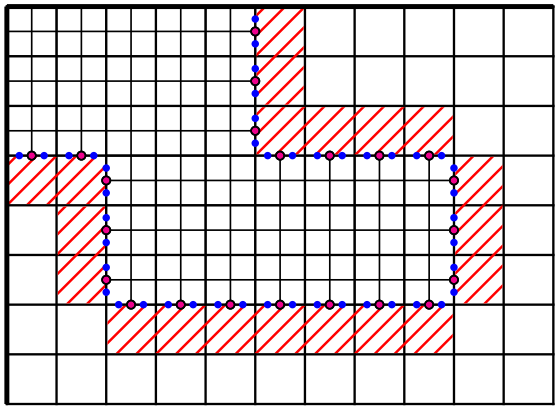
Correction pass:

1. $\delta\mathbf{F}^{1,l+1}_{j-\frac{1}{2},k} := -\mathbf{F}^{1,l}_{j-\frac{1}{2},k}$
2. $\delta\mathbf{F}^{1,l+1}_{j-\frac{1}{2},k} := \delta\mathbf{F}^{1,l+1}_{j-\frac{1}{2},k} + \frac{1}{r^2_{l+1}}\sum_{\iota=0}^{r_{l+1}-1}\mathbf{F}^{1,l+1}_{v+\frac{1}{2},w+\iota}(t+\kappa\Delta t_{l+1})$
3. $\check{\mathbf{Q}}^l_{jk}(t+\Delta t_l) := \mathbf{Q}^l_{jk}(t+\Delta t_l) + \frac{\Delta t_l}{\Delta x_{1,l}}\delta\mathbf{F}^{1,l+1}_{j-\frac{1}{2},k}$



Conservative flux correction II

- Level l cells needing correction $(G^{r_{l+1}}_{l+1} \backslash G_{l+1}) \cap G_l$
- Corrections $\delta\mathbf{F}^{n,l+1}$ stored on level $l+1$ along ∂G_{l+1} (lower-dimensional data coarsened by r_{l+1})
- Init $\delta\mathbf{F}^{n,l+1}$ with level l fluxes $\mathbf{F}^{n,l}(\bar{G}_l \cap \partial G_{l+1})$
- Add level $l+1$ fluxes $\mathbf{F}^{n,l+1}(\partial G_{l+1})$ to $\delta\mathbf{F}^{n,l}$

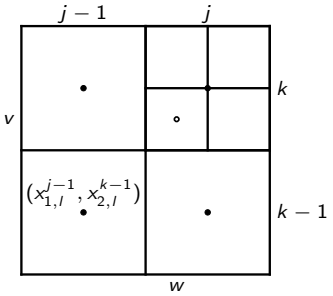


Cells to correct • $\mathbf{F}^{n,l}$ • $\mathbf{F}^{n,l+1}$ ∘ $\delta\mathbf{F}^{n,l+1}$

Level transfer operators

Conservative averaging (restriction):
Replace cells on level l covered by level $l+1$, i.e. $G_l \cap G_{l+1}$, by

$$\hat{\mathbf{Q}}^l_{jk} := \frac{1}{(r_{l+1})^2} \sum_{\kappa=0}^{r_{l+1}-1} \sum_{\iota=0}^{r_{l+1}-1} \mathbf{Q}^{l+1}_{v+\kappa,w+\iota}$$



Bilinear interpolation (prolongation):

$$\check{\mathbf{Q}}^{l+1}_{vw} := (1-f_1)(1-f_2)\,\mathbf{Q}^l_{j-1,k-1} + f_1(1-f_2)\,\mathbf{Q}^l_{j,k-1} + (1-f_1)f_2\,\mathbf{Q}^l_{j-1,k} + f_1 f_2\,\mathbf{Q}^l_{jk}$$

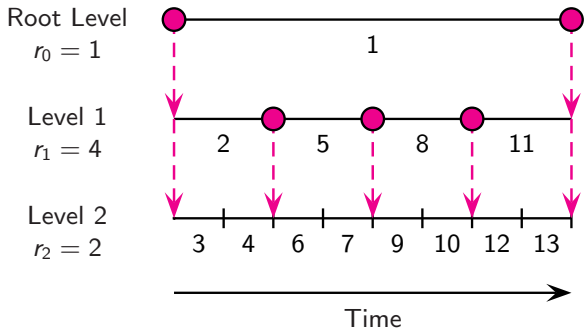
with factors $f_1 := \dfrac{x^v_{1,l+1} - x^{j-1}_{1,l}}{\Delta x_{1,l}}$, $f_2 := \dfrac{x^w_{2,l+1} - x^{k-1}_{2,l}}{\Delta x_{2,l}}$ derived from the spatial coordinates of the cell centers $(x^{j-1}_{1,l}, x^{k-1}_{2,l})$ and $(x^v_{1,l+1}, x^w_{2,l+1})$.

For boundary conditions on $\tilde{l}^s_l$: linear time interpolation

$$\tilde{\mathbf{Q}}^{l+1}(t+\kappa\Delta t_{l+1}) := \left(1 - \frac{\kappa}{r_{l+1}}\right) \check{\mathbf{Q}}^{l+1}(t) + \frac{\kappa}{r_{l+1}} \check{\mathbf{Q}}^{l+1}(t+\Delta t_l) \quad \text{for } \kappa = 0, \ldots r_{l+1}$$

Recursive integration order

- Space-time interpolation of coarse data to set l^s_l, $l > 0$
- Regridding:
  - Creation of new grids, copy existing cells on level $l > 0$
  - Spatial interpolation to initialize new cells on level $l > 0$



Regridding of finer levels.
Base level (●) stays fixed.

The basic recursive algorithm

```
AdvanceLevel(l)

  Repeat r_l times
      Set ghost cells of Q^l(t)
      If time to regrid?
          Regrid(l)
      UpdateLevel(l)
      If level l+1 exists?
          Set ghost cells of Q^l(t + Δt_l)
          AdvanceLevel(l + 1)
          Average Q^{l+1}(t + Δt_l) onto Q^l(t + Δt_l)
          Correct Q^l(t + Δt_l) with δF^{l+1}
      t := t + Δt_l

Start - Start integration on level 0

    l = 0, r_0 = 1
    AdvanceLevel(l)
```

- Recursion
- Restriction and flux correction
- Re-organization of hierarchical data

[Berger and Colella, 1988][Berger and Oliger, 1984]

Regridding algorithm

```
Regrid(l) - Regrid all levels ι > l

  For ι = l_f Downto l Do
      Flag N^ι according to Q^ι(t)
      If level ι + 1 exists?
          Flag N^ι below Ğ^{ι+2}
      Flag buffer zone on N^ι
      Generate Ğ^{ι+1} from N^ι
  Ğ_l := G_l
  For ι = l To l_f Do
      CĞ_ι := G_0 \ Ğ_ι
      Ğ_{ι+1} := Ğ_{ι+1} \ CĞ^1_ι

  Recompose(l)
```

- Refinement flags: $N^l := \bigcup_m N(\partial G_{l,m})$
- Activate flags below higher levels
- Flag buffer cells of $b > \kappa_r$ cells, κ_r steps between calls of `Regrid(l)`
- Special cluster algorithm
- Use complement operation to ensure proper nesting condition

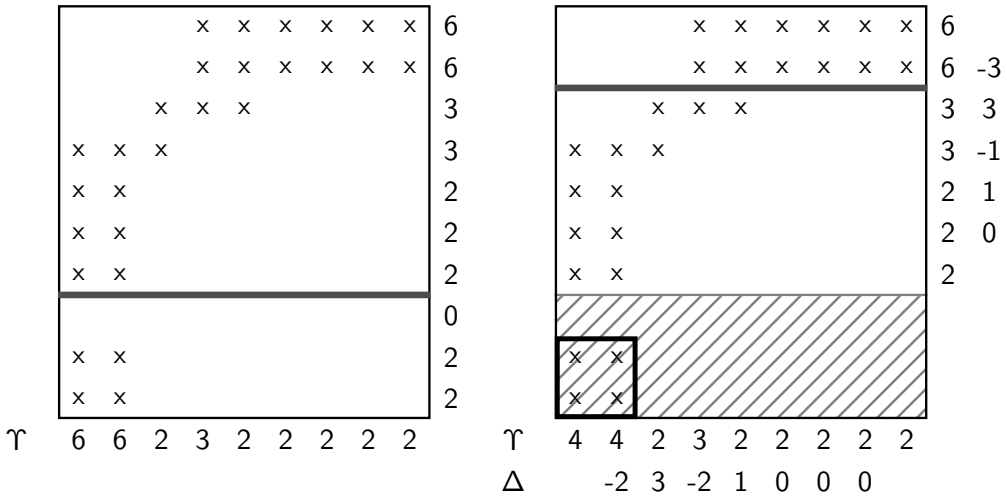
Recomposition of data

`Recompose(`l`)` - Reorganize all levels $\iota > l$

```
For ι = l + 1 To l_f + 1 Do
    Interpolate Q^{ι-1}(t) onto Q̆^ι(t)
    Copy Q^ι(t) onto Q̆^ι(t)
    Set ghost cells of Q̆^ι(t)
    Q^ι(t) := Q̆^ι(t),  G_ι := Ğ_ι
```

- Creates max. 1 level above l_f, but can remove multiple level if $\breve{G}_\iota$ empty (no coarsening!)
- Use spatial interpolation on entire data $\breve{\mathbf{Q}}^\iota(t)$
- Overwrite where old data exists
- Synchronization and physical boundary conditions

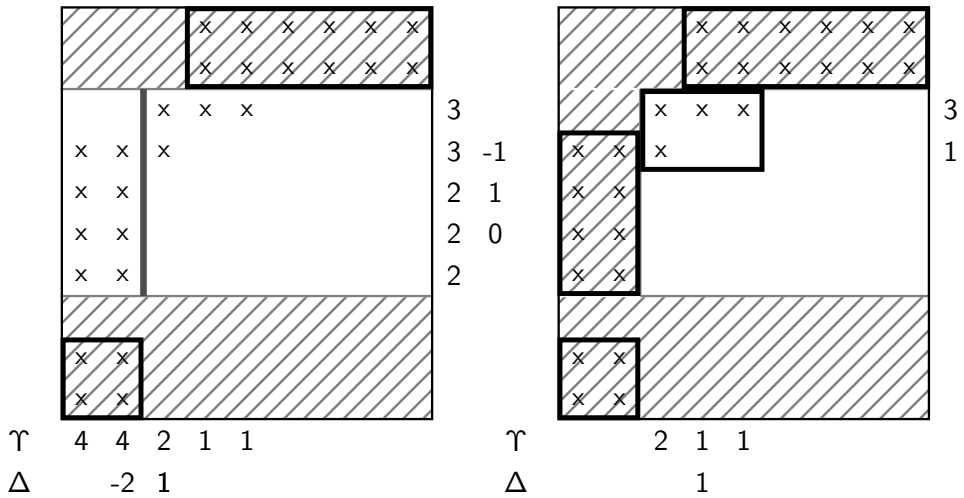
Clustering by signatures



ϒ Flagged cells per row/column

Δ Second derivative of ϒ, $\Delta = \Upsilon_{\nu+1} - 2\,\Upsilon_\nu + \Upsilon_{\nu-1}$

Technique from image detection: [Bell et al., 1994], see also [Berger and Rigoutsos, 1991], [Berger, 1986]



Recursive generation of $\breve{G}_{l,m}$

1. 0 in ϒ
2. Largest difference in Δ
3. Stop if ratio between flagged and unflagged cell $> \eta_{tol}$

Refinement criteria

Scaled gradient of scalar quantity w

$$|w(\mathbf{Q}_{j+1,k}) - w(\mathbf{Q}_{jk})| > \epsilon_w\,,\ |w(\mathbf{Q}_{j,k+1}) - w(\mathbf{Q}_{jk})| > \epsilon_w\,,\ |w(\mathbf{Q}_{j+1,k+1}) - w(\mathbf{Q}_{jk})| > \epsilon_w$$

Heuristic error estimation [Berger, 1982]:
Local truncation error of scheme of order o

$$\mathbf{q}(\mathbf{x}, t+\Delta t) - \mathcal{H}^{(\Delta t)}(\mathbf{q}(\cdot, t)) = \mathbf{C}\Delta t^{o+1} + O(\Delta t^{o+2})$$

For $\mathbf{q}$ smooth after 2 steps Δt

$$\mathbf{q}(\mathbf{x}, t+\Delta t) - \mathcal{H}_2^{(\Delta t)}(\mathbf{q}(\cdot, t-\Delta t)) = 2\,\mathbf{C}\Delta t^{o+1} + O(\Delta t^{o+2})$$

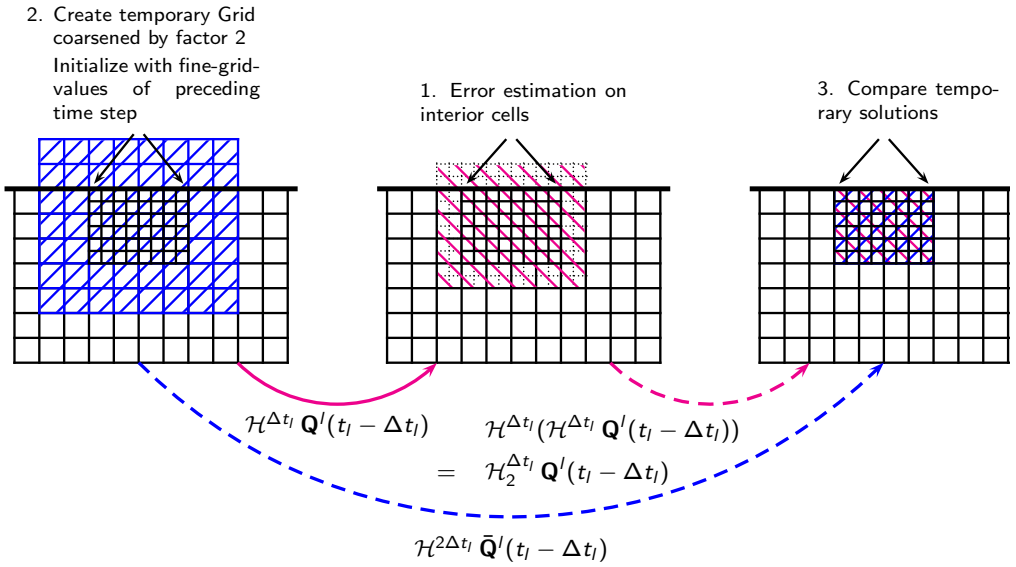
and after 1 step with $2\Delta t$

$$\mathbf{q}(\mathbf{x}, t+\Delta t) - \mathcal{H}^{(2\Delta t)}(\mathbf{q}(\cdot, t-\Delta t)) = 2^{o+1}\mathbf{C}\Delta t^{o+1} + O(\Delta t^{o+2})$$

Gives

$$\mathcal{H}_2^{(\Delta t)}(\mathbf{q}(\cdot, t-\Delta t)) - \mathcal{H}^{(2\Delta t)}(\mathbf{q}(\cdot, t-\Delta t)) = (2^{o+1}-2)\mathbf{C}\Delta t^{o+1} + O(\Delta t^{o+2})$$

Heuristic error estimation for FV methods



Usage of heuristic error estimation

Current solution integrated tentatively 1 step with Δt_l and coarsened

$$\bar{\mathcal{Q}}(t_l+\Delta t_l) := \text{Restrict}\left(\mathcal{H}_2^{\Delta t_l}\,\mathbf{Q}^l(t_l-\Delta t_l)\right)$$

Previous solution coarsened and integrated 1 step with $2\Delta t_l$

$$\mathcal{Q}(t_l+\Delta t_l) := \mathcal{H}^{2\Delta t_l}\,\text{Restrict}\left(\mathbf{Q}^l(t_l-\Delta t_l)\right)$$

Local error estimation of scalar quantity w

$$\tau_{jk}^w := \frac{|w(\bar{\mathcal{Q}}_{jk}(t+\Delta t)) - w(\mathcal{Q}_{jk}(t+\Delta t))|}{2^{o+1}-2}$$

In practice [Deiterding, 2003] use

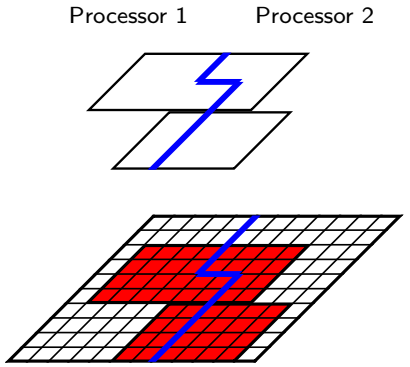
$$\frac{\tau_{jk}^w}{\max(|w(\mathcal{Q}_{jk}(t+\Delta t))|, S_w)} > \eta_w^r$$

Outline

Parallelization strategies

Decomposition of the hierarchical data

- Distribution of each grid
- Separate distribution of each level, cf. [Rendleman et al., 2000]
- Rigorous domain decomposition
  - Data of all levels resides on same node
  - Grid hierarchy defines unique "floor-plan"
  - Redistribution of data blocks during reorganization of hierarchical data
  - Synchronization when setting ghost cells



Rigorous domain decomposition formalized

Parallel machine with P identical nodes. P non-overlapping portions G_0^p, $p = 1, \ldots, P$ as

$$G_0 = \bigcup_{p=1}^{P} G_0^p \quad \text{with} \quad G_0^p \cap G_0^q = \emptyset \ \text{ for } p \neq q$$

Higher level domains G_l follow decomposition of root level

$$G_l^p := G_l \cap G_0^p$$

With $\mathcal{N}_l(\cdot)$ denoting number of cells, we estimate the workload as

$$\mathcal{W}(\Omega) = \sum_{l=0}^{l_{\max}} \left[\mathcal{N}_l(G_l \cap \Omega) \prod_{\kappa=0}^{l} r_\kappa \right]$$

Equal work distribution necessitates

$$\mathcal{L}^p := \frac{P \cdot \mathcal{W}(G_0^p)}{\mathcal{W}(G_0)} \approx 1 \quad \text{for all } p = 1, \ldots, P$$

[Deiterding, 2005]

Ghost cell setting

Local synchronization

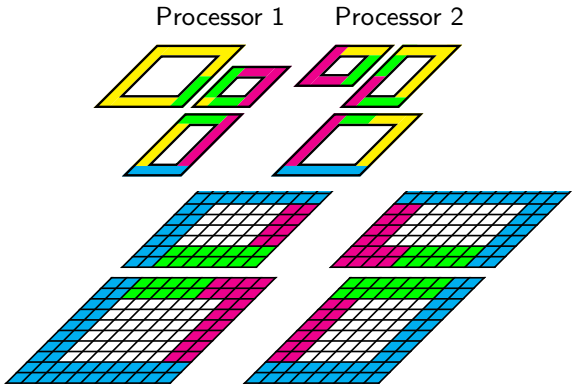
$$\tilde{S}_{l,m}^{s,p} = \tilde{G}_{l,m}^{s,p} \cap G_l^p$$

Parallel synchronization

$$\tilde{S}_{l,m}^{s,q} = \tilde{G}_{l,m}^{s,p} \cap G_l^q, q \neq p$$

Interpolation and physical boundary conditions remain strictly local

- Scheme $\mathcal{H}^{(\Delta t_l)}$ evaluated locally
- Restriction and propolongation local

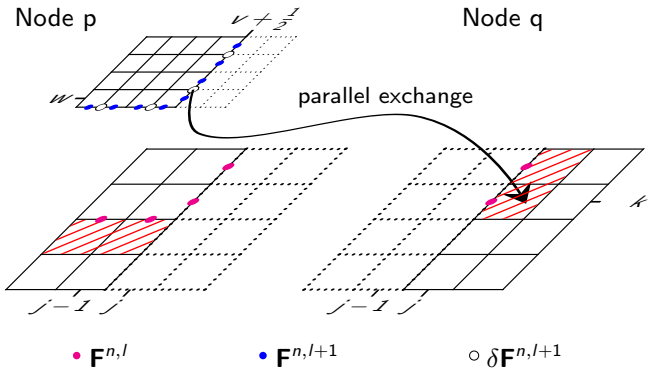


Ghost cell values:

Interpolation | Parallel synchronization
Local synchronization | Physical boundary

Parallel flux correction

1. Strictly local: Init $\delta\mathbf{F}^{n,l+1}$ with $\mathbf{F}^n(\bar{G}_{l,m} \cap \partial G_{l+1}, t)$
2. Strictly local: Add $\mathbf{F}^n(\partial G_{l,m}, t)$ to $\delta\mathbf{F}^{n,l}$
3. Parallel communication: Correct $\mathbf{Q}^l(t + \Delta t_l)$ with $\delta\mathbf{F}^{l+1}$



The recursive algorithm in parallel

```
AdvanceLevel(l)

  Repeat r_l times
      Set ghost cells of Q^l(t)
      If time to regrid?
          Regrid(l)
      UpdateLevel(l)
      If level l+1 exists?
          Set ghost cells of Q^l(t + Δt_l)
          AdvanceLevel(l+1)
          Average Q^{l+1}(t + Δt_l) onto Q^l(t + Δt_l)
          Correct Q^l(t + Δt_l) with δF^{l+1}
      t := t + Δt_l

UpdateLevel(l)

  For all m = 1 To M_l Do
      Q(G^s_{l,m}, t) --H^{(Δt_l)}--> Q(G_{l,m}, t + Δt_l), F^n(Ḡ_{l,m}, t)
      If level l > 0
          Add F^n(∂G_{l,m}, t) to δF^{n,l}
      If level l+1 exists
          Init δF^{n,l+1} with F^n(Ḡ_{l,m} ∩ ∂G_{l+1}, t)
```

- Numerical update strictly local
- Inter-level transfer local
- Parallel synchronization
- Application of $\delta\mathbf{F}^{l+1}$ on ∂G_l^q

Regridding algorithm in parallel

```
Regrid(l) - Regrid all levels ι > l
  For ι = l_f Downto l Do
      Flag N^ι according to Q^ι(t)
      If level ι+1 exists?
          Flag N^ι below Ğ^{ι+2}
      Flag buffer zone on N^ι
      Generate Ğ^{ι+1} from N^ι
  Ğ_l := G_l
  For ι = l To l_f Do
      CĞ_ι := G_0 \ Ğ_ι
      Ğ_{ι+1} := Ğ_{ι+1} \ CĞ^1_ι
  Recompose(l)
```

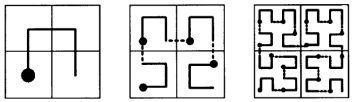
- Need a ghost cell overlap of b cells to ensure correct setting of refinement flags in parallel
- Two options exist (we choose the latter):
  - Global clustering algorithm
  - Local clustering algorithm and concatenation of new lists $\check{G}^{\iota+1}$

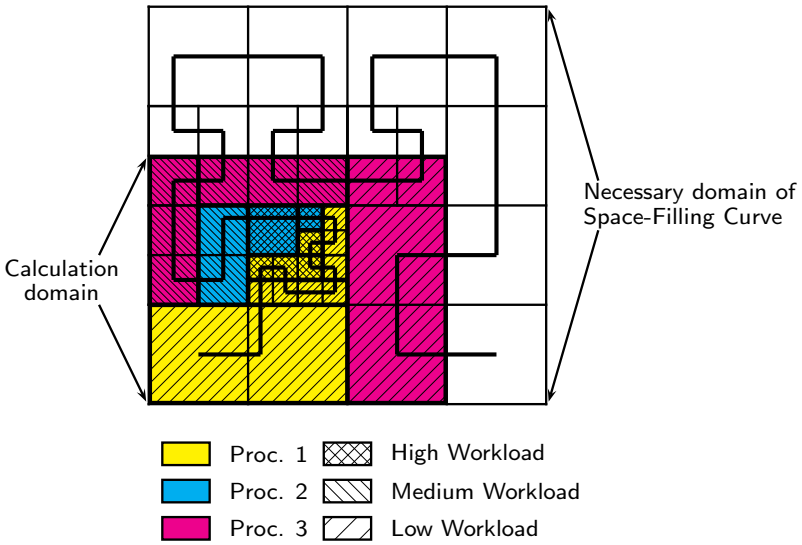
Recomposition algorithm in parallel

```
Recompose(l) - Reorganize all levels
  Generate G_0^p from {G_0, ..., G_l, Ğ_{l+1}, ..., Ğ_{l_f+1}}
  For ι = l+1 To l_f + 1 Do
      If ι > l
          Ğ_ι^p := Ğ_ι ∩ G_0^p
          Interpolate Q^{ι-1}(t) onto Q̆^ι(t)
      else
          Ğ_ι^p := G_ι ∩ G_0^p
          If ι > 0
              Copy δF^{n,ι} onto δF̆^{n,ι}
              δF^{n,ι} := δF̆^{n,ι}
      If ι ≥ l then κ_ι = 0 else κ_ι = 1
      For κ = 0 To κ_ι Do
          Copy Q^ι(t) onto Q̆^ι(t)
          Set ghost cells of Q̆^ι(t)
          Q^ι(t) := Q̆^ι(t)
      G_ι := Ğ_ι
```

- Global redistribution can also be required when regridding higher levels and $G_0, ..., G_l$ do not change (drawback of domain decomposition)
- When $\iota > l$ do nothing special
- For $\iota \leq l$, redistribute additionally
  - Flux corrections $\delta\mathbf{F}^{n,\iota}$
  - Already updated time level $\mathbf{Q}^\iota(t + \kappa\Delta t_\iota)$

Space-filling curve algorithm



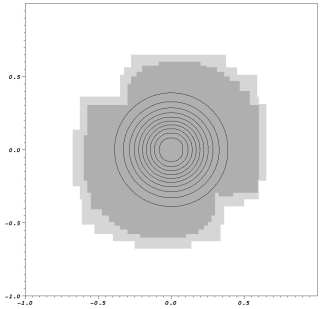


Outline

SAMR accuracy verification

Gaussian density shape

$$\rho(x_1, x_2) = 1 + e^{-\left(\frac{\sqrt{x_1^2+x_2^2}}{R}\right)^2}$$



is advected with constant velocities $u_1 = u_2 \equiv 1$, $p_0 \equiv 1$, $R = 1/4$

- Domain $[-1,1] \times [-1,1]$, periodic boundary conditions, $t_{end} = 2$
- Two levels of adaptation with $r_{1,2} = 2$, finest level corresponds to $N \times N$ uniform grid

Use *locally* conservative interpolation

$$\check{\mathbf{Q}}^l_{v,w} := \mathbf{Q}^l_{ij} + f_1(\mathbf{Q}^l_{i+1,j} - \mathbf{Q}^l_{i-1,j}) + f_2(\mathbf{Q}^l_{i,j+1} - \mathbf{Q}^l_{i,j-1})$$

with factor $f_1 = \dfrac{x^v_{1,l+1} - x^i_{1,l}}{2\Delta x_{1,l}}$, $f_2 = \dfrac{x^w_{2,l+1} - x^j_{2,l}}{2\Delta x_{2,l}}$ to also test flux correction

This prolongation operator is not monotonicity preserving! Only applicable to smooth problems.

SAMR accuracy verification: results

VanLeer flux vector splitting with dimensional splitting, Minmod limiter

N	Unigrid		SAMR - fixup			SAMR - no fixup		
	Error	Order	Error	Order	$\Delta\rho$	Error	Order	$\Delta\rho$
20	0.10946400							
40	0.04239430	1.369						
80	0.01408160	1.590	0.01594820		0	0.01595980		2e-5
160	0.00492945	1.514	0.00526693	1.598	0	0.00530538	1.589	2e-5
320	0.00146132	1.754	0.00156516	1.751	0	0.00163837	1.695	-1e-5
640	0.00041809	1.805	0.00051513	1.603	0	0.00060021	1.449	-6e-5

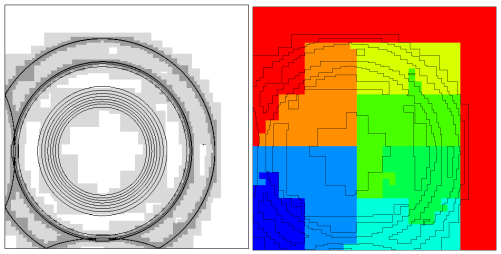
Fully two-dimensional Wave Propagation Method, Minmod limiter

N	Unigrid		SAMR - fixup			SAMR - no fixup		
	Error	Order	Error	Order	$\Delta\rho$	Error	Order	$\Delta\rho$
20	0.10620000							
40	0.04079600	1.380						
80	0.01348250	1.598	0.01536580		0	0.01538820		2e-5
160	0.00472301	1.513	0.00505406	1.604	0	0.00510499	1.592	5e-5
320	0.00139611	1.758	0.00147218	1.779	0	0.00152387	1.744	7e-5
640	0.00039904	1.807	0.00044500	1.726	0	0.00046587	1.710	6e-5

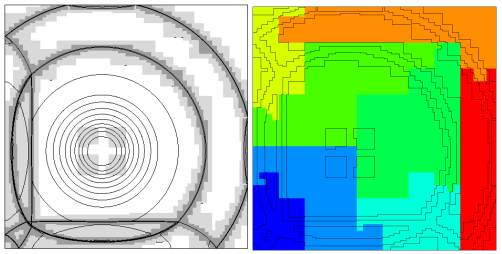
Benchmark run: blast wave in 2D

- 2D-Wave-Propagation Method with Roe's approximate solver
- Base grid 150×150
- 2 levels: factor 2, 4

Task [%]	P=1	P=2	P=4	P=8	P=16
Update by $\mathcal{H}^{(\cdot)}$	86.6	83.4	76.7	64.1	51.9
Flux correction	1.2	1.6	3.0	7.9	10.7
Boundary setting	3.5	5.7	10.1	15.6	18.3
Recomposition	5.5	6.1	7.4	9.9	14.0
Misc.	4.9	3.2	2.8	2.5	5.1
Time [min]	**151.9**	**79.2**	**43.4**	**23.3**	**13.9**
Efficiency [%]	**100.0**	**95.9**	**87.5**	**81.5**	**68.3**



After 38 time steps

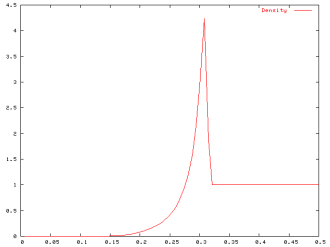


After 79 time steps

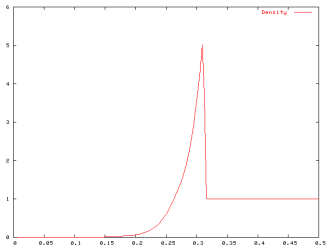
Benchmark run 2: point-explosion in 3D

- Benchmark from the Chicago workshop on AMR methods, September 2003
- Sedov explosion - energy deposition in sphere of radius 4 finest cells
- 3D-Wave-Prop. Method with hybrid Roe-HLL scheme
- Base grid 32^3
- Refinement factor $r_l = 2$
- Effective resolutions: 128^3, 256^3, 512^3, 1024^3
- Grid generation efficiency $\eta_{tol} = 85\%$
- Proper nesting enforced
- Buffer of 1 cell

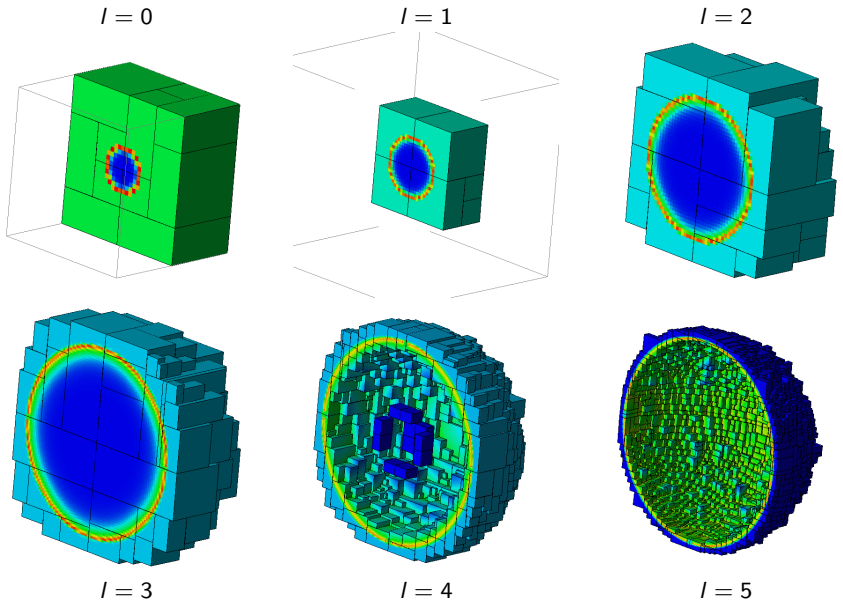
$l_{\max} = 4$ solution



$l_{\max} = 5$ solution



Benchmark run 2: visualization of refinement



Benchmark run 2: performance results

Number of grids and cells

l	$l_{max} = 2$		$l_{max} = 3$		$l_{max} = 4$		$l_{max} = 5$	
	Grids	Cells	Grids	Cells	Grids	Cells	Grids	Cells
0	28	32,768	28	32,768	33	32,768	34	32,768
1	8	32,768	14	32,768	20	32,768	20	32,768
2	63	115,408	49	116,920	43	125,680	50	125,144
3			324	398,112	420	555,744	193	572,768
4					1405	1,487,312	1,498	1,795,048
5							5,266	5,871,128
$\sum$		180,944		580,568		2,234,272		8,429,624

Breakdown of CPU time on 8 nodes SGI Altix 3000 (Linux-based shared memory system)

Task [%]	$l_{max} = 2$		$l_{max} = 3$		$l_{max} = 4$		$l_{max} = 5$	
Integration	73.7		77.2		72.9		37.8	
Fixup	2.6	*46*	3.1	*58*	2.6	*42*	2.2	*45*
Boundary	10.1	*79*	6.3	*78*	5.1	*56*	6.9	*78*
Recomposition	7.4		8.0		15.1		50.4	
Clustering	0.5		0.6		0.7		1.0	
Output/Misc	5.7		4.0		3.6		1.7	
Time [min]	0.5		5.1		73.0		2100.0	
Uniform [min]	5.4		160		∼5,000		∼180,000	
Factor of AMR savings	11		31		69		86	
Time steps	15		27		52		115	

References I

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Lecture 3

Complex hyperbolic applications

Course *Block-structured Adaptive Mesh Refinement Methods for Conservation Laws*
Theory, Implementation and Application

Ralf Deiterding
Computer Science and Mathematics Division
Oak Ridge National Laboratory
P.O. Box 2008 MS6367, Oak Ridge, TN 37831, USA

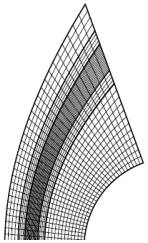
E-mail: deiterdingr@ornl.gov

Outline

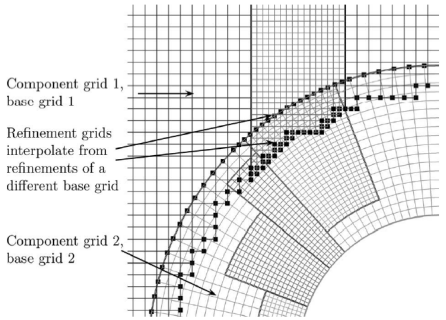
Outline

SAMR on boundary aligned meshes

Analytic or stored geometric mapping of the coordinates (graphic from [Yamaleev and Carpenter, 2002])



- Topology remains unchanged and thereby entire SAMR algorithm
- Patch solver and interpolation need to consider geometry transformation
- Handles boundary layers well



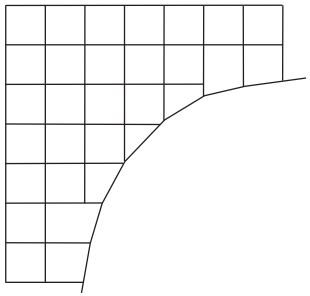
Overlapping adaptive meshes [Henshaw and Schwendeman, 2003], [Meakin, 1995]

- Idea is to use a non-Cartesian structured grids only near boundary
- Very suitable for moving objects with boundary layers
- Interpolation between meshes is usually non-conservative

Cut-cell techniques

Accurate embedded boundary method

$$V_j^{n+1}\mathbf{Q}_j^{n+1} = V_j^n\mathbf{Q}_j^n - \Delta t\left(A_{j+1/2}^{n+1/2}\,\mathbf{F}(\mathbf{Q},j) - A_{j-1/2}^{n+1/2}\,\mathbf{F}(\mathbf{Q},j-1)\right)$$



Methods that represent the boundary sharply:

- Cut-cell approach constructs appropriate finite volumes
- Conservative by construction. Correct boundary flux
- Key question: How to avoid small-cell time step restriction in explicit metyhods?
  - Cell merging: [Quirk, 1994a]
- Usually explicit geometry representation used [Aftosmis, 1997], but can also be implicit, cf. [Nourgaliev et al., 2003], [Murman et al., 2003]

Embedded boundary techniques

Volume of fluid methods that resemble a cut-cell technique on purely Cartesian mesh

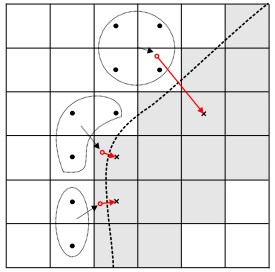
- Redistribution of boundary flux achieves conservation and bypasses time step restriction: [Pember et al., 1999], [Berger and Helzel, 2002]

Methods that diffuse the boundary in one cell (good overview in [Mittal and Iaccarino, 2005]):

- Related to the immersed boundary method by Peskin, cf. [Roma et al., 1999]
- Boundary prescription often by internal ghost cell values, cf. [Tseng and Ferziger, 2003]
- Not conservative by construction but conservative correction possible
- Usually combined with implicit geometry representation

K. J. Richards et al., On the use of the immersed boundary method for engine modeling

Level-set method for boundary embedding



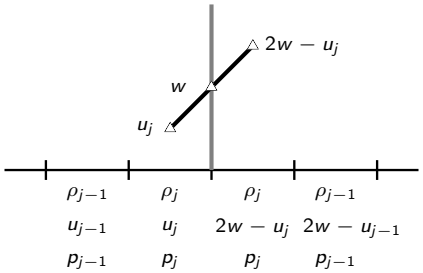
- Implicit boundary representation via distance function φ, normal $\mathbf{n} = \nabla\varphi/|\nabla\varphi|$
- Complex boundary moving with local velocity $\mathbf{w}$, treat interface as moving rigid wall
- Construction of values in embedded boundary cells by interpolation / extrapolation

Interpolate / constant value extrapolate values at

$$\tilde{\mathbf{x}} = \mathbf{x} + 2\varphi\mathbf{n}$$

Velocity in ghost cells

$$\begin{aligned}\mathbf{u}' &= (2\mathbf{w}\cdot\mathbf{n} - \mathbf{u}\cdot\mathbf{n})\mathbf{n} + (\mathbf{u}\cdot\mathbf{t})\mathbf{t} \\ &= 2\left((\mathbf{w}-\mathbf{u})\cdot\mathbf{n}\right)\mathbf{n} + \mathbf{u}\end{aligned}$$



Closest point transform algorithm

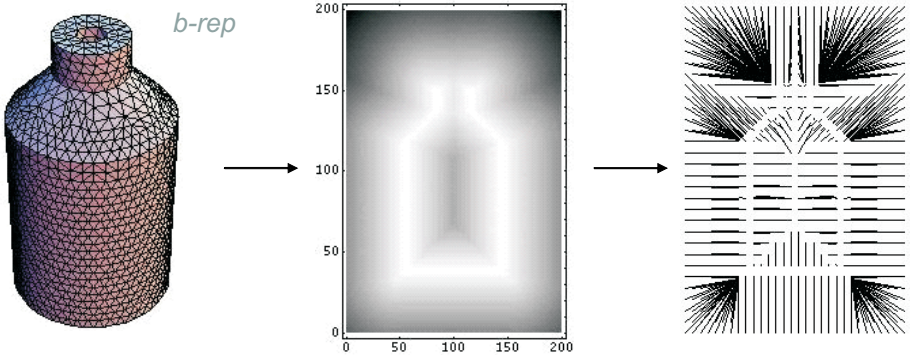
The signed distance φ to a surface $\mathcal{I}$ satisfies the eikonal equation [Sethian, 1999]

$$|\nabla\varphi| = 1 \quad \text{with} \quad \varphi\big|_{\mathcal{I}} = 0$$

Solution smooth but non-diferentiable across characteristics.

Distance computation trivial for non-overlapping elementary shapes but difficult to do efficiently for triangulated surface meshes:

- Geometric solution approach with plosest-point-transform algorithm [Mauch, 2003]



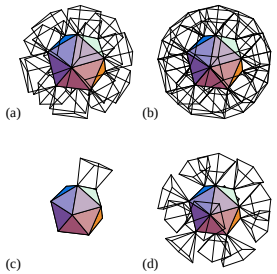
Surface mesh $\mathcal{I}$ | Distance φ | Normal to closest point

The characteristic / scan conversion algorithm

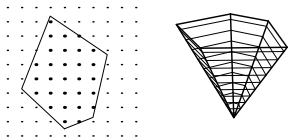
1. Build the characteristic polyhedrons for the surface mesh
2. For each face/edge/vertex
   - 2.1 Scan convert the polyhedron.
   - 2.2 Compute distance to that primitive for the scan converted points
3. Computational complexity.
   - $O(m)$ to build the b-rep and the polyhedra.
   - $O(n)$ to scan convert the polyhedra and compute the distance, etc.
4. Problem reduction by evaluation only within specified max. distance

[Mauch, 2003], see also [Deiterding et al., 2006]

Characteristic polyhedra for faces, edges, and vertices



Slicing and scan conversion of apolygon



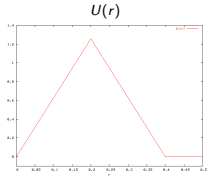
Accuracy test: stationary vortex

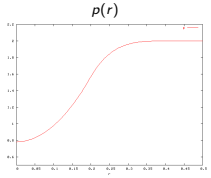
Construct non-trivial *radially symmetric* and *stationary* solution by balancing hydrodynamic pressure and centripetal force per volume element, i.e.

$$\frac{d}{dr}p(r) = \rho(r)\frac{U(r)^2}{r}$$

For $\rho_0 \equiv 1$ and the velocity field

$$U(r) = \alpha \cdot \begin{cases} 2r/R & \text{if } 0 < r < R/2, \\ 2(1 - r/R) & \text{if } R/2 \le r \le R, \\ 0 & \text{if } r > R, \end{cases}$$





one gets with boundary condition $p(R) = p_0 \equiv 2$ the pressure distribution

$$p(r) = p_0 + 2\rho_0\alpha^2 \cdot \begin{cases} r^2/R^2 + 1 - 2\log 2 & \text{if } 0 < r < R/2, \\ r^2/R^2 + 3 - 4r/R + 2\log(r/R) & \text{if } R/2 \le r \le R, \\ 0 & \text{if } r > R. \end{cases}$$

Entire solution for Euler equations reads

$$\rho(x_1, x_2, t) = \rho_0,\ u_1(x_1, x_2, t) = -U(r)\sin\phi,\ u_2(x_1, x_2, t) = U(r)\cos\phi,\ p(x_1, x_2, t) = p(r)$$

for all $t \ge 0$ with $r = \sqrt{(x_1 - x_{1,c})^2 + (x_2 - x_{2,c})^2}$ and $\phi = \arctan\dfrac{x_2 - x_{2,c}}{x_1 - x_{1,c}}$

Stationary vortex: results

Compute one full rotation, Roe solver, embedded slip wall boundary conditions
$x_{1,c} = 0.5,\ x_{2,c} = 0.5,\ R = 0.4,\ t_{end} = 1,\ \Delta h = \Delta x_1 = \Delta x_2 = 1/N,\ \alpha = R\pi$

No embedded boundary

N	Wave Propagation		Godunov Splitting	
	Error	Order	Error	Order
20	0.0111235		0.0182218	
40	0.0037996	1.55	0.0090662	1.01
80	0.0013388	1.50	0.0046392	0.97
160	0.0005005	1.42	0.0023142	1.00

Marginal shear flow along embedded boundary, $\alpha = R\pi,\ R_G = R,\ U_W = 0$

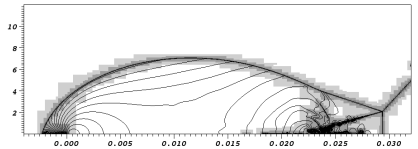
N	Wave Propagation			Godunov Splitting		
	Error	Order	Mass loss	Error	Order	Mass loss
20	0.0120056		0.0079236	0.0144203		0.0020241
40	0.0035074	1.78	0.0011898	0.0073070	0.98	0.0001300
80	0.0014193	1.31	0.0001588	0.0038401	0.93	-0.0001036
160	0.0005032	1.50	5.046e-05	0.0018988	1.02	-2.783e-06

Major shear flow along embedded boundary, $\alpha = R\pi,\ R_G = R/2,\ U_W = 0$

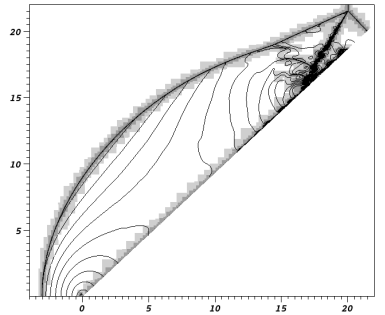
N	Wave Propagation			Godunov Splitting		
	Error	Order	Mass loss	Error	Order	Mass loss
20	0.0423925		0.0423925	0.0271446		0.0271446
40	0.0358735	0.24	0.0358735	0.0242260	0.16	0.0242260
80	0.0212340	0.76	0.0212340	0.0128638	0.91	0.0128638
160	0.0121089	0.81	0.0121089	0.0070906	0.86	0.0070906

Verification: shock reflection

- Reflection of a Mach 2.38 shock in nitrogen at 43° wedge
- 2nd order MUSCL scheme with Roe solver, 2nd order multidimensional wave propagation method

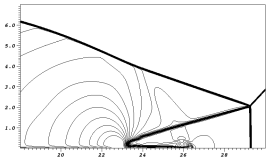


Cartesian base grid 360 × 160 cells on domain of 36 mm × 16 mm with up to 3 refinement levels with $r_l = 2, 4, 4$ and $\Delta x_{1,2} = 3.125\mu m$, 38 h CPU

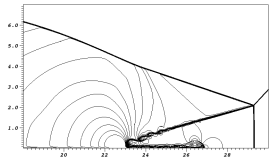


GFM base grid 390 × 330 cells on domain of 26 mm × 22 mm with up to 3 refinement levels with $r_l = 2, 4, 4$ and $\Delta x_{e,1,2} = 2.849\mu m$, 200 h CPU

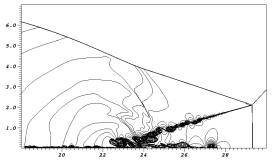
Shock reflection: SAMR solution for Euler equations



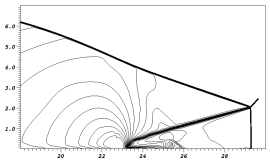
$\Delta x = 25$ mm



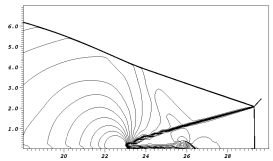
$\Delta x = 12.5$ mm



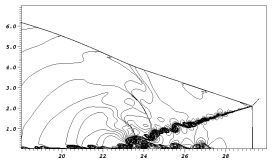
$\Delta x = 3.125$ mm



$\Delta x_e = 22.8$ mm

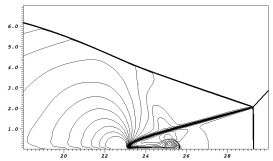


$\Delta x_e = 11.4$ mm

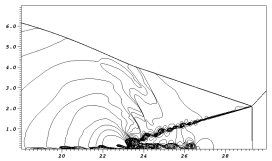


$\Delta x_e = 2.849$ mm

2nd order MUSCL scheme with Van Leer FVS, dimensional splitting



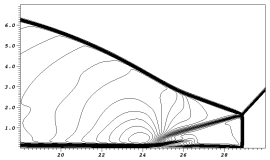
$\Delta x = 12.5$ mm



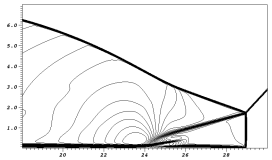
$\Delta x = 3.125$ mm

Shock reflection: solution for Navier-Stokes equations

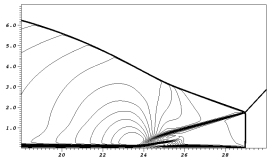
- No-slip boundary conditions enforced
- Conservative 2nd order centered differences to approximate stress tensor and heat flow



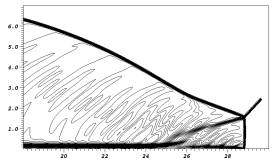
$\Delta x = 50$ mm



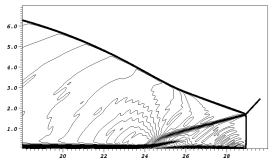
$\Delta x = 25$ mm



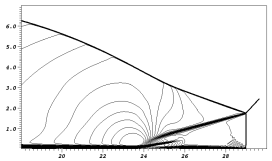
$\Delta x = 12.5$ mm, SAMR



$\Delta x_e = 45.6$ mm



$\Delta x_e = 22.8$ mm



$\Delta x_e = 11.4$ mm, SAMR

Outline

Governing equations for premixed combustion

Euler equations with reaction terms

$$\frac{\partial \rho_i}{\partial t} + \frac{\partial}{\partial x_n}(\rho_i u_n) = \dot{\omega}_i \,, \quad i = 1, \ldots, K$$

$$\frac{\partial}{\partial t}(\rho u_k) + \frac{\partial}{\partial x_n}(\rho u_k u_n + \delta_{kn} p) = 0 \,, \quad k = 1, \ldots, d$$

$$\frac{\partial}{\partial t}(\rho E) + \frac{\partial}{\partial x_n}(u_n(\rho E + p)) = 0$$

Ideal gas law and Dalton's law for gas-mixtures

$$p(\rho_1, \ldots, \rho_K, T) = \sum_{i=1}^{K} p_i = \sum_{i=1}^{K} \rho_i \frac{\mathcal{R}}{W_i} T = \rho \frac{\mathcal{R}}{W} T \quad \text{with} \quad \sum_{i=1}^{K} \rho_i = \rho \,, Y_i = \frac{\rho_i}{\rho}$$

Caloric equation

$$h(Y_1, \ldots, Y_K, T) = \sum_{i=1}^{K} Y_i h_i(T) \quad \text{with} \quad h_i(T) = h_i^0 + \int_0^T c_{pi}(s) ds$$

Computation of $T = T(\rho_1, \ldots, \rho_K, e)$ from implicit equation

$$\sum_{i=1}^{K} \rho_i \, h_i(T) - \mathcal{R} T \sum_{i=1}^{K} \frac{\rho_i}{W_i} - \rho e = 0$$

for *thermally perfect* gases with $\gamma_i(T) = c_{pi}(T)/c_{vi}(T)$

Chemistry

Arrhenius-Kinetics:

$$\dot{\omega}_i = \sum_{j=1}^{M} (\nu''_{ji} - \nu'_{ji}) \left[k_j^f \prod_{n=1}^{K} \left(\frac{\rho_n}{W_n} \right)^{\nu'_{jn}} - k_j^r \prod_{n=1}^{K} \left(\frac{\rho_n}{W_n} \right)^{\nu''_{jn}} \right] \quad i = 1, \ldots, K$$

- Parsing of mechanisms with Chemkin-II
- Evalutation of $\dot{\omega}_i$ with automatically generated optimized Fortran-77 functions in the line of Chemkin-II

Integration of reaction rates: ODE integration in $\mathcal{S}^{(\cdot)}$ for Euler equations with chemical reaction

- Standard implicit or semi-implicit ODE-solver subcycles within each cell
- ρ, e, u_k remain unchanged!

$$\partial_t \rho_i = W_i \dot{\omega}_i(\rho_1, \ldots, \rho_K, T) \qquad i = 1, \ldots, K$$

Use Newton or bisection method to compute T iteratively.

Non-equilibrium mechanism for hydrogen-oxgen combustion

				A [cm, mol, s]	β	E_{act} [cal mol^{-1}]
1.	$H + O_2$	$\longrightarrow$	$O + OH$	1.86×10^{14}	0.00	16790.
2.	$O + OH$	$\longrightarrow$	$H + O_2$	1.48×10^{13}	0.00	680.
3.	$H_2 + O$	$\longrightarrow$	$H + OH$	1.82×10^{10}	1.00	8900.
4.	$H + OH$	$\longrightarrow$	$H_2 + O$	8.32×10^{09}	1.00	6950.
5.	$H_2O + O$	$\longrightarrow$	$OH + OH$	3.39×10^{13}	0.00	18350.
6.	$OH + OH$	$\longrightarrow$	$H_2O + O$	3.16×10^{12}	0.00	1100.
7.	$H_2O + H$	$\longrightarrow$	$H_2 + OH$	9.55×10^{13}	0.00	20300.
8.	$H_2 + OH$	$\longrightarrow$	$H_2O + H$	2.19×10^{13}	0.00	5150.
9.	$H_2O_2 + OH$	$\longrightarrow$	$H_2O + HO_2$	1.00×10^{13}	0.00	1800.
10.	$H_2O + HO_2$	$\longrightarrow$	$H_2O_2 + OH$	2.82×10^{13}	0.00	32790.
...	...		...	...	...	...
30.	$OH + M$	$\longrightarrow$	$O + H + M$	7.94×10^{19}	-1.00	103720.
31.	$O_2 + M$	$\longrightarrow$	$O + O + M$	5.13×10^{15}	0.00	115000.
32.	$O + O + M$	$\longrightarrow$	$O_2 + M$	4.68×10^{15}	-0.28	0.
33.	$H_2 + M$	$\longrightarrow$	$H + H + M$	2.19×10^{14}	0.00	96000.
34.	$H + H + M$	$\longrightarrow$	$H_2 + M$	3.02×10^{15}	0.00	0.

Third body efficiencies: $f(O_2) = 0.40$, $f(H_2O) = 6.50$

C. K. Westbrook. Chemical kinetics of hydrocarbon oxidation in gaseous detonations. *J. Combustion and Flame*, 46:191–210, 1982.

Riemann solver for combustion

(S1) Calculate standard Roe-averages $\hat{\rho}$, $\hat{u}_n$, $\hat{H}$, $\hat{Y}_i$, $\hat{T}$.

(S2) Compute $\hat{\gamma} := \hat{c}_p / \hat{c}_v$ with $\hat{c}_{\{p/v\}i} = \dfrac{1}{T_R - T_L} \displaystyle\int_{T_L}^{T_R} c_{\{p,v\}i}(\tau)\, d\tau$.

(S3) Calculate $\hat{\phi}_i := (\hat{\gamma} - 1) \left(\frac{\hat{\mathbf{u}}^2}{2} - \hat{h}_i \right) + \hat{\gamma} R_i \hat{T}$ with standard Roe-averages $\hat{e}_i$ or $\hat{h}_i$.

(S4) Calculate $\hat{c} := \left(\sum_{i=1}^{K} \hat{Y}_i \hat{\phi}_i - (\hat{\gamma} - 1)\hat{\mathbf{u}}^2 + (\hat{\gamma} - 1)\hat{H} \right)^{1/2}$.

(S5) Use $\Delta \mathbf{q} = \mathbf{q}_R - \mathbf{q}_L$ and Δp to compute the wave strengths a_m.

(S6) Calculate $\mathcal{W}_1 = a_1 \hat{\mathbf{r}}_1$, $\mathcal{W}_2 = \displaystyle\sum_{\iota=2}^{K+d} a_\iota \hat{\mathbf{r}}_\iota$, $\mathcal{W}_3 = a_{K+d+1} \hat{\mathbf{r}}_{K+d+1}$.

(S7) Evaluate $s_1 = \hat{u}_1 - \hat{c}$, $s_2 = \hat{u}_1$, $s_3 = \hat{u}_1 + \hat{c}$.

(S8) Evaluate $\rho^\star_{L/R}$, $u^\star_{1,L/R}$, $e^\star_{L/R}$, $c^\star_{1,L/R}$ from $\mathbf{q}^\star_L = \mathbf{q}_L + \mathcal{W}_1$ and $\mathbf{q}^\star_R = \mathbf{q}_R - \mathcal{W}_3$.

(S9) If $\rho^\star_{L/R} \leq 0$ or $e^\star_{L/R} \leq 0$ use $\mathbf{F}_{HLL}(\mathbf{q}_L, \mathbf{q}_R)$ and go to (S12).

(S10) Entropy correction: Evaluate $|\tilde{s}_\iota|$.
$\mathbf{F}_{Roe}(\mathbf{q}_L, \mathbf{q}_R) = \frac{1}{2} \left(\mathbf{f}(\mathbf{q}_L) + \mathbf{f}(\mathbf{q}_R) - \sum_{\iota=1}^{3} |\tilde{s}_\iota| \mathcal{W}_\iota \right)$

(S11) Positivity correction: Replace $\mathbf{F}_i$ by
$\mathbf{F}^\star_i = \mathbf{F}_\rho \cdot \begin{cases} Y^l_i\,, & \mathbf{F}_\rho \geq 0\,, \\ Y^r_i\,, & \mathbf{F}_\rho < 0\,. \end{cases}$

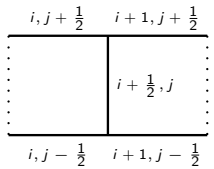
(S12) Evaluate maximal signal speed by $S = \max(|s_1|, |s_3|)$.

Riemann solver for combustion: carbuncle fix

Entropy corrections [Harten, 1983] [Harten and Hyman, 1983]

1. $|\tilde{s}_\iota| = \begin{cases} |s_\iota| & \text{if} |s_\iota| \geq 2\eta \\ \frac{|s_\iota^2|}{4\eta} + \eta & \text{otherwise} \end{cases}$
   $\eta = \frac{1}{2} \max_\iota \left\{ |s_\iota(\mathbf{q}_R) - s_\iota(\mathbf{q}_L)| \right\}$
2. Replace $|s_\iota|$ by $|\tilde{s}_\iota|$ only if $s_\iota(\mathbf{q}_L) < 0 < s_\iota(\mathbf{q}_R)$

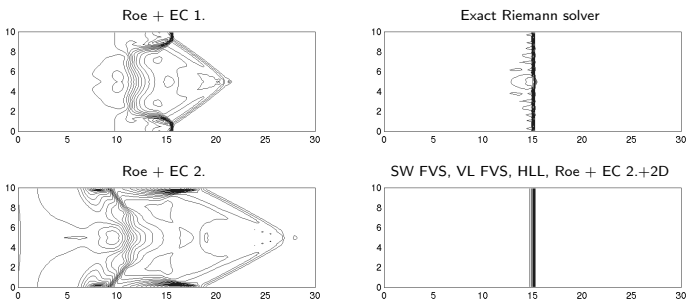
2D modification of entropy correction [Sanders et al., 1998]:



$$\tilde{\eta}_{i+1/2,j} = \max \left\{ \eta_{i+1/2,j}, \eta_{i,j-1/2}, \eta_{i,j+1/2}, \eta_{i+1,j-1/2}, \eta_{i+1,j+1/2} \right\}$$

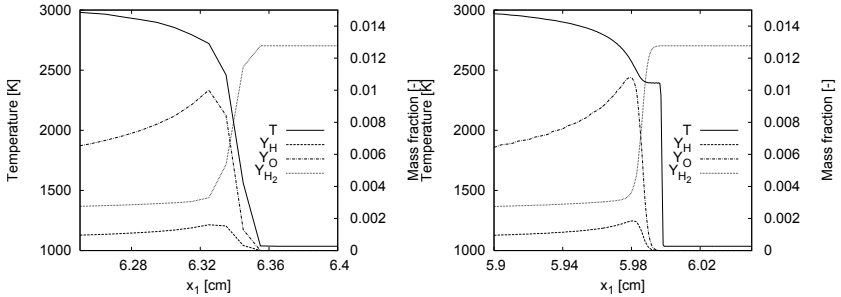
Carbuncle phenomenon

- [Quirk, 1994b]
- Test from [Deiterding, 2003]



Detonations - motivation for SAMR

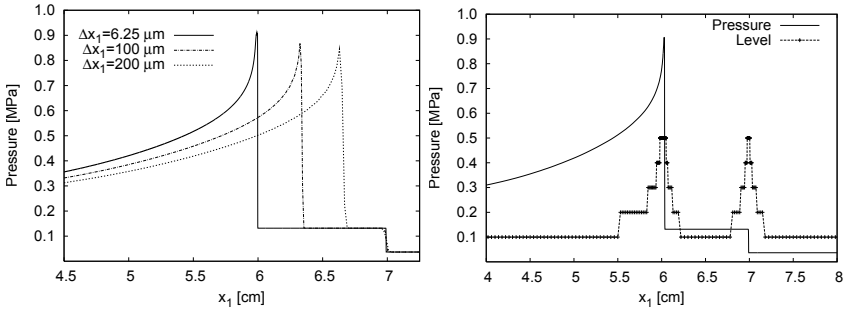
- Extremly high spatial resolution in reaction zone necessary.
- Minimal spatial resolution: $7 - 8\,\mathrm{Pts}/l_{ig} \longrightarrow \Delta x_1 \approx 0.2 - 0.175\,\mathrm{mm}$
- Uniform grids for typical geometries: $> 10^7\,\mathrm{Pts}$ in 2D, $> 10^9\,\mathrm{Pts}$ in 3D $\longrightarrow$ Self-adaptive finite volume method (AMR)



Approximation of $H_2 : O_2$ detonation at $\sim 1.5\,\mathrm{Pts}/l_{ig}$ (left) and $\sim 24\,\mathrm{Pts}/l_{ig}$ (right)

Detonation ignition in a shock tube

- Shock-induced detonation ignition of $H_2 : O_2 : Ar$ mixture at molar ratios 2:1:7 in closed 1d shock tube
- Insufficient resolution leads to inaccurate results
- Reflected shock is captured correctly by FV scheme, detonation is resolution dependent
- Fine mesh necessary in the induction zone at the head of the detonation



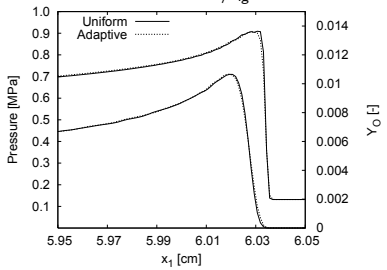
Left: Comparison of pressure distribution $t = 170\,\mu\mathrm{s}$ after shock reflection.Right: Domains of refinement levels

Detonation ignition in 1d - adaptive vs. uniform

Uniformly refined vs. dynamic adaptive simulations (Intel Xeon 3.4 GHz CPU)

	Uniform			Adaptive			
$\Delta x_1[\mu m]$	Cells	$t_m[\mu s]$	Time [s]	l_{max}	r_l	$t_m[\mu s]$	Time [s]
400	300	166.1	31				
200	600	172.6	90	2	2	172.6	99
100	1200	175.5	277	3	2,2	175.8	167
50	2400	176.9	858	4	2,2,2	177.3	287
25	4800	177.8	2713	4	2,2,4	177.9	393
12.5	9600	178.3	9472	5	2,2,2,4	178.3	696
6.25	19200	178.6	35712	5	2,2,4,4	178.6	1370

$\sim 12\,\mathrm{Pts}/l_{ig}$



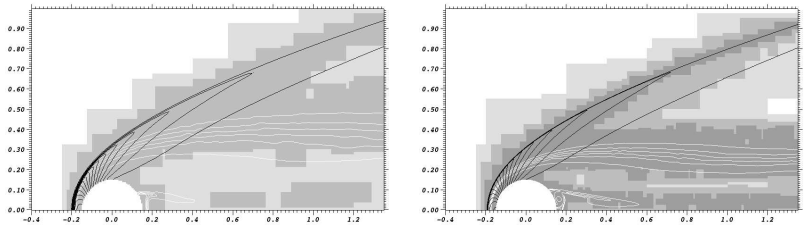
Refinement criteria:

Y_i	$S_{Y_i} \cdot 10^{-4}$	$\eta^r_{Y_i} \cdot 10^{-3}$
O_2	10.0	2.0
H_2O	7.8	8.0
H	0.16	5.0
O	1.0	5.0
OH	1.8	5.0
H_2	1.3	2.0

$\epsilon_\rho = 0.07\,\mathrm{kg\,m^{-3}}$, $\epsilon_p = 50\,\mathrm{kPa}$

Shock-induced combustion around a sphere

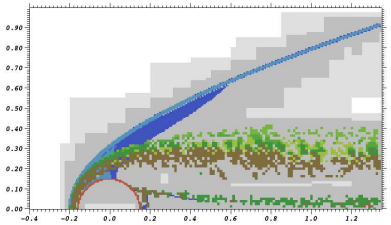
- Spherical projectile of radius 1.5 mm travels with constant velocity $v_l = 2170.6\,\mathrm{m/s}$ through $H_2 : O_2 : Ar$ mixture (molar ratios 2:1:7) at 6.67 kPa and $T = 298\,\mathrm{K}$
- Cylindrical symmetric simulation on AMR base mesh of 70×40 cells
- Comparison of 3-level computation with refinement factors 2,2 ($\sim 5\,\mathrm{Pts}/l_{ig}$) and a 4-level computation with refinement factors 2,2,4 ($\sim 19\,\mathrm{Pts}/l_{ig}$) at $t = 350\,\mu\mathrm{s}$
- Higher resolved computation captures combustion zone visibly better and at slightly different position (see below)



Iso-contours of p (black) and Y_{H_2} (white) on refinement domains for 3-level (left) and 4-level computation (right)

Combustion around a sphere - adaptation

Refinement indicators on $l = 2$ at $t = 350\,\mu$s. Blue: ϵ_ρ, light blue: ϵ_p, green shades: $\eta^r_{Y_i}$, red: embedded boundary

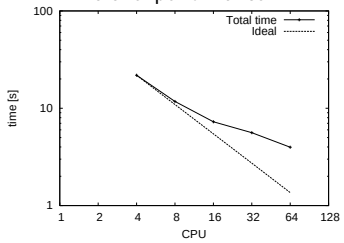


Refinement criteria:

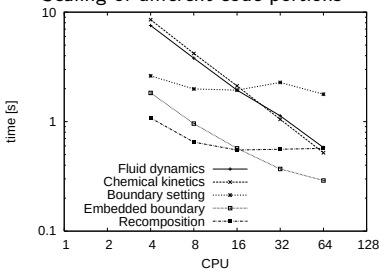
Y_i	$S_{Y_i} \cdot 10^{-4}$	$\eta^r_{Y_i} \cdot 10^{-4}$
O_2	10.0	4.0
H_2O	5.8	3.0
H	0.2	10.0
O	1.4	10.0
OH	2.3	10.0
H_2	1.3	4.0

$\epsilon_\rho = 0.02\,\mathrm{kg\,m^{-3}}$, $\epsilon_p = 16\,\mathrm{kPa}$

Parallel performance

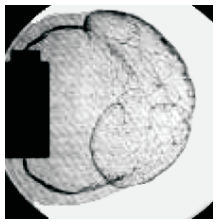


Scaling of different code portions

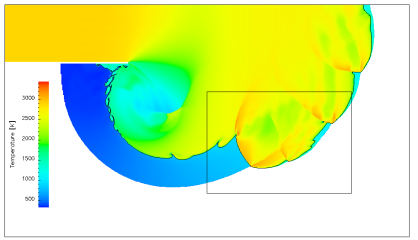


Detonation diffraction

- CJ detonation for $H_2 : O_2 : Ar/2 : 1 : 7$ at $T_0 = 298\,\mathrm{K}$ and $p_0 = 10\,\mathrm{kPa}$. Cell width $\lambda_c = 1.6\,\mathrm{cm}$
- Adaption criteria (similar as before):
  1. Scaled gradients of ρ and p
  2. Error estimation in Y_i by Richardson extrapolation
- $25\,\mathrm{Pts}/l_{ig}$. 5 refinement levels (2,2,2,4).
- Adaptive computations use up to $\sim 2.2\,\mathrm{M}$ instead of $\sim 150\,\mathrm{M}$ cells (uniform grid)
- $\sim 3850\,\mathrm{h}$ CPU ($\sim 80\,\mathrm{h}$ real time) on 48 nodes Athlon 1.4GHz

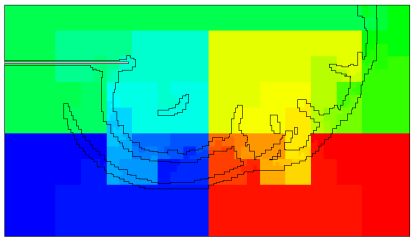


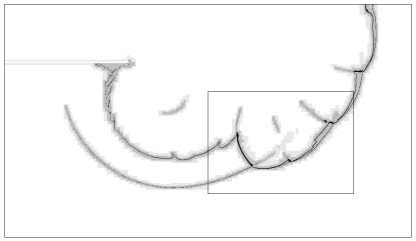
E. Schultz. *Detonation diffraction through an abrupt area expansion*. PhD thesis, California Institute of Technology, Pasadena, California, April 2000.

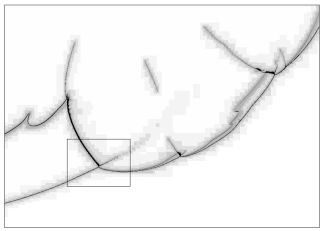


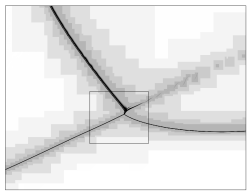
Detonation diffraction - adaptation

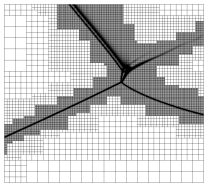
Final distribution to 48 nodes and density distribution on four refinement levels







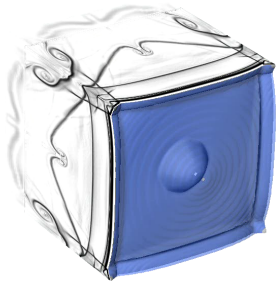




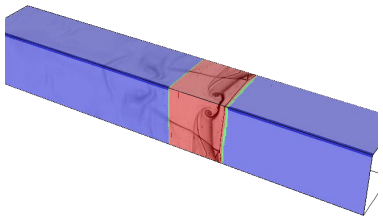
Detonation cell structure in 3D

- Simulation of only one quadrant
- $44.8\,\mathrm{Pts}/l_{ig}$ for $H_2 : O_2 : Ar$ CJ detonation
- SAMR base grid 400x24x24, 2 additional refinement levels (2, 4)
- Simulation uses $\sim 18\,\mathrm{M}$ cells instead of $\sim 118\,\mathrm{M}$ (unigrid)
- $\sim 51,000\,\mathrm{h}$ CPU on 128 CPU Compaq Alpha. $\mathcal{H}$: 37.6 %, $\mathcal{S}$: 25.1 %

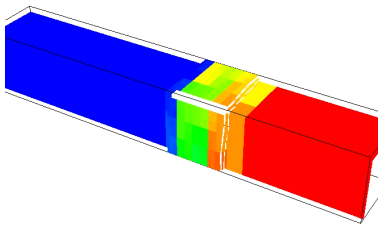
Schlieren and isosurface of Y_{OH}



Schlieren on refinement levels



Distribution to 128 processors

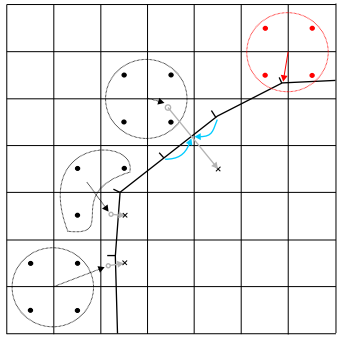


Outline

Construction of coupling data

- Moving boundary/interface is treated as a moving contact discontinuity and represented by level set [Fedkiw, 2002][Arienti et al., 2003]
- One-sided construction of mirrored ghost cell and new FEM nodal point values
- FEM ansatz-function interpolation to obtain intermediate surface values
- Explicit coupling possible if geometry and velocities are prescribed for the more compressible medium [Specht, 2000]

$u_n^F := u_n^S(t)|_{\mathcal{I}}$
`UpdateFluid(`Δt`)`
$\sigma_{nn}^S := p^F(t+\Delta t)|_{\mathcal{I}}$
`UpdateSolid(`Δt`)`
$t := t + \Delta t$



Coupling conditions on interface

$$\left.\begin{array}{rcl} u_n^S &=& u_n^F \\ \sigma_{nn}^S &=& p^F \\ \sigma_{nm}^S &=& 0 \end{array}\right|_{\mathcal{I}}$$

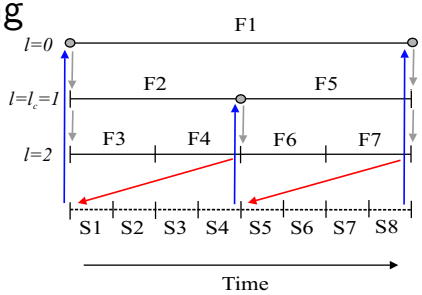
Usage of SAMR

- Eulerian SAMR + non-adaptive Lagrangian FEM scheme
- Exploit SAMR time step refinement for effective coupling to solid solver
  - Lagrangian simulation is called only at level $l_c \leq l_{\max}$
  - SAMR refines solid boundary at least at level l_c
  - Additional levels can be used resolve geometric ambiguities
- Nevertheless: Inserting sub-steps accommodates for time step reduction from the solid solver within an SAMR cycle
- Communication strategy:
  - Updated boundary info from solid solver must be received before regridding operation
  - Boundary data is sent to solid when highest level available
- Inter-solver communication (point-to-point or globally) managed on the fly special coupling module

SAMR algorithm for FSI coupling

```
AdvanceLevel(l)

 Repeat r_l times
   Set ghost cells of Q^l(t)
   CPT(φ^l, C^l, I, δ_l)
   If time to regrid?
       Regrid(l)
   UpdateLevel(l)
   If level l+1 exists?
       Set ghost cells of Q^l(t+Δt_l)
       AdvanceLevel(l+1)
       Average Q^{l+1}(t+Δt_l) onto Q^l(t+Δt_l)
   If l = l_c?
       SendInterfaceData(p^F(t+Δt_l)|_I)
       If (t+Δt_l) < (t_0+Δt_0)?
           ReceiveInterfaceData(I, u^S|_I)
   t := t + Δt_l
```



- Call CPT algorithm before `Regrid(l)`
- Include also call to `CPT(·)` into `Recompose(l)` to ensure consistent level set data on levels that have changed
- Communicate boundary data on coupling level l_c

Fluid and solid update / exchange of time steps

`FluidStep( )`

$\Delta\tau_F := \min_{l=0,\cdots,l_{\max}} (R_l \cdot$ `StableFluidTimeStep(`l`)`, $\Delta\tau_S)$
$\Delta t_l := \Delta\tau_F / R_l$ `for` $l = 0, \cdots, L$
`ReceiveInterfaceData(`$\mathcal{I}$, $\mathbf{u}^S|_{\mathcal{I}}$`)`
`AdvanceLevel(0)`

`SolidStep( )`

$\Delta\tau_S := \min(K \cdot R_{l_c} \cdot$ `StableSolidTimeStep()`, $\Delta\tau_F)$
`Repeat` R_{l_c} `times`
  $t_{\text{end}} := t + \Delta\tau_S / R_{l_c}$, $\Delta t := \Delta\tau_S / (K R_{l_c})$
  `While` $t < t_{\text{end}}$
    `SendInterfaceData(`$\mathcal{I}(t)$, $\vec{u}^S|_{\mathcal{I}}(t)$`)`
    `ReceiveInterfaceData(`$p^F|_{\mathcal{I}}$`)`
    `UpdateSolid(`$p^F|_{\mathcal{I}}$, Δt`)`
    $t := t + \Delta t$
    $\Delta t := \min($`StableSolidTimeStep()`, $t_{\text{end}} - t)$

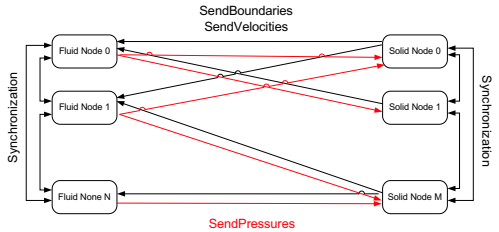
with $R_l = \prod_{\iota=0}^{l} r_\iota$

- Time step stays constant for R_{l_c} steps, which correponds to one fluid step at level 0

Parallelization strategy for coupled simulations

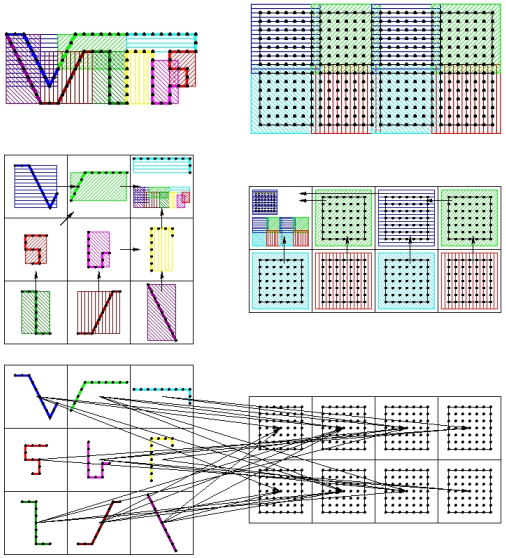
Coupling of an Eulerian FV fluid Solver and a Lagrangian FEM Solver:

- Distribute both meshes seperately and copy necessary nodal values and geometry data to fluid nodes
- Setting of ghost cell values becomes strictly local operation
- Construct new nodal values strictly local on fluid nodes and transfer them back to solid nodes
- Only surface data is transfered
- Asynchronous communication ensures scalability
- Generic encapsulated implementation guarantees reusability



Eulerian/Lagrangian communication module

1. Put bounding boxes around each solid processors piece of the boundary and around each fluid processors grid
2. Gather, exchange and broadcast of bounding box information
3. Optimal point-to-point communication pattern, non-blocking



Lift-up of a spherical body

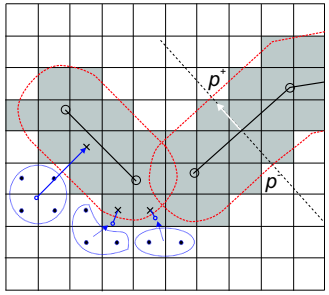
Cylindrical body hit by Mach 3 shockwave, 2D test case by [Falcovitz et al., 1997]

Schlieren plot of density

Refinement levels

Treatment of thin structures

- Thin boundary structures or lower-dimensional shells require "thickening" to apply embedded boundary method
- Unsigned distance level set function φ
- Treat cells with $0 < \varphi < d$ as ghost fluid cells



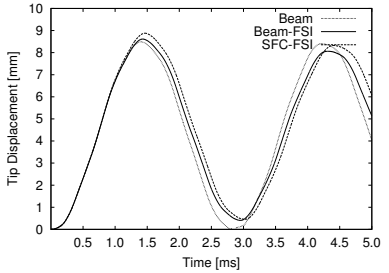
- Leaving φ unmodified ensures correctness of $\nabla\varphi$
- Use face normal in shell element to evaluate in $\Delta p = p^+ - p^-$
- Utilize finite difference solver using the beam equation

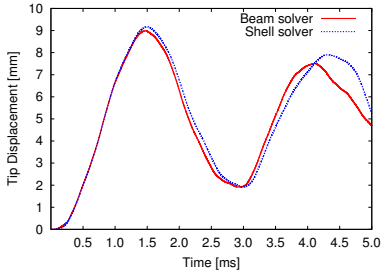
$$\rho_s h \frac{\partial^2 w}{\partial t^2} + EI \frac{\partial^4 w}{\partial \bar{x}^4} = p^F$$

to verify FSI algorithms

FSI verification by elastic vibration

- Thin steel plate (thickness $h = 1\,\mathrm{mm}$, length $50\,\mathrm{mm}$), clamped at lower end
- $\rho_s = 7600\,\mathrm{kg/m^3}$, $E = 220\,\mathrm{GPa}$, $I = h^3/12$, $\nu = 0.3$
- Modeled with beam solver (101 points) and thin-shell FEM solver (325 triangles) by F. Cirak
- Left: Coupling verification with constant instantenous loading by $\Delta p = 100\,\mathrm{kPa}$
- Right: FSI verification with Mach 1.21 shockwave in air ($\gamma = 1.4$)

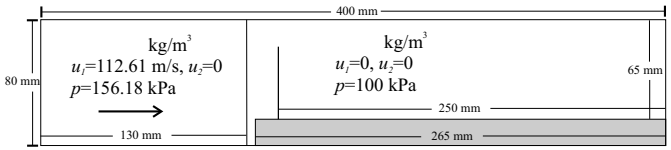




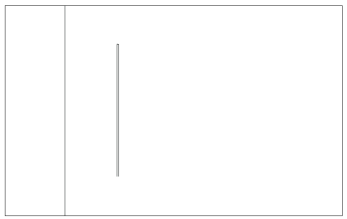
Shock-driven elastic panel motion

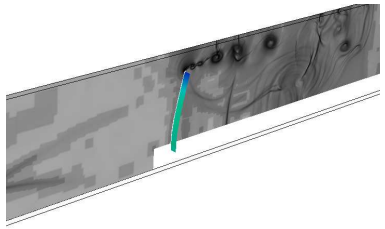
Test case suggested by [Giordano et al., 2005]

- Forward facing step geometry, fixed walls everywhere except at inflow



- SAMR base mesh $320 \times 64(\times 2)$, $r_{1,2} = 2$
- Intel 3.4GHz Xeon dual processors, GB Ethernet interconnect
  - Beam-FSI: 12.25 h CPU on 3 fluid CPU + 1 solid CPU
  - FEM-FSI: 322 h CPU on 14 fluid CPU + 2 solid CPU





$t = 1.56\,\mathrm{ms}$ after impact

Detonation-driven plastic deformation

Chapman-Jouguet detonation in a tube filled with a stoichiometric ethylene and oxygen ($C_2H_4 + 3\,O_2$, $295\,\mathrm{K}$) mixture. Euler equations with single exothermic reaction $A \longrightarrow B$

$$\partial_t \rho + \partial_{x_n}(\rho u_n) = 0\,, \quad \partial_t(\rho u_k) + \partial_{x_n}(\rho u_k u_n + \delta_{kn} p) = 0\,, k = 1, \ldots, d$$

$$\partial_t(\rho E) + \partial_{x_n}(u_n(\rho E + p)) = 0\,, \quad \partial_t(Y\rho) + \partial_{x_n}(Y\rho u_n) = \psi$$

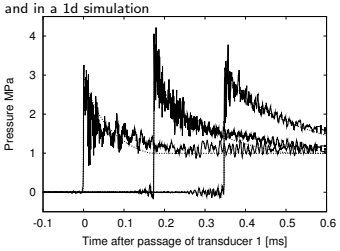
with

$$p = (\gamma - 1)(\rho E - \frac{1}{2}\rho u_n u_n - \rho Y q_0) \quad \text{and} \quad \psi = -kY\rho \exp\left(\frac{-E_A \rho}{p}\right)$$

modeled with heuristic detonation model by [Mader, 1979]

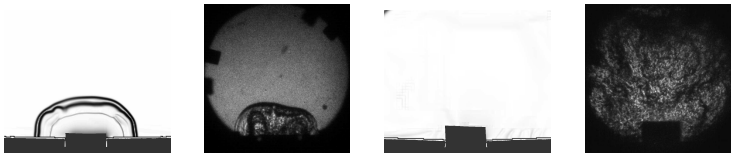
$V := \rho^{-1}$, $V_0 := \rho_0^{-1}$, $V_{CJ} := \rho_{CJ}$
$Y' := 1 - (V - V_0)/(V_{CJ} - V_0)$
If $0 \le Y' \le 1$ and $Y > 10^{-8}$ then
  If $Y < Y'$ and $Y' < 0.9$ then $Y' := 0$
  If $Y' < 0.99$ then $p' := (1 - Y')p_{CJ}$
    else $p' := p$
  $\rho_A := Y'\rho$
  $E := p'/(\rho(\gamma - 1)) + Y' q_0 + \frac{1}{2} u_n u_n$

Comparison of the pressure traces in the experiment and in a 1d simulation

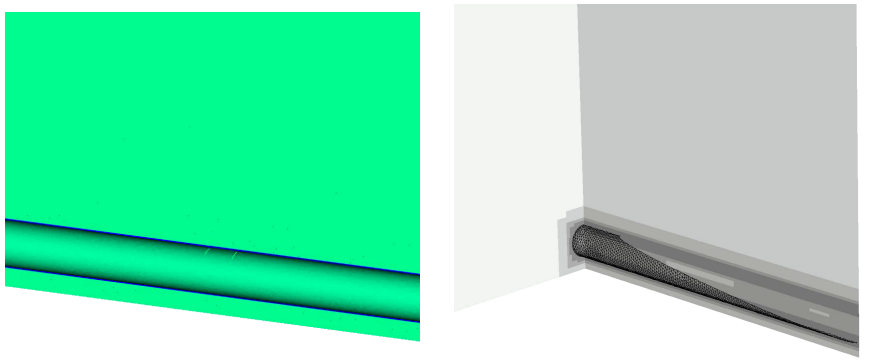


Tube with flaps

- Fluid: VanLeer FVS
  - Detonation model with $\gamma = 1.24$, $p_{\mathrm{CJ}} = 3.3\,\mathrm{MPa}$, $D_{\mathrm{CJ}} = 2376\,\mathrm{m/s}$
  - AMR base level: $104 \times 80 \times 242$, $r_{1,2} = 2$, $r_3 = 4$
  - $\sim 4 \cdot 10^7$ cells instead of $7.9 \cdot 10^9$ cells (uniform)
  - Tube and detonation fully refined
  - Thickening of 2D mesh: $0.81\,\mathrm{mm}$ on both sides (real $0.445\,\mathrm{mm}$)
- Solid: thin-shell solver by F. Cirak
  - Aluminum, J2 plasticity with hardening, rate sensitivity, and thermal softening
  - Mesh: 8577 nodes, 17056 elements
- 16+2 nodes 2.2 GHz AMD Opteron quad processor, PCI-X 4x Infiniband network, $\sim 4320\,\mathrm{h}$ CPU to $t_{end} = 450\,\mu\mathrm{s}$



Tube with flaps: results



Fluid density and diplacement in y-direction in solid

Schlieren plot of fluid density on refinement levels

[Cirak et al., 2007]

Underwater explosion modeling

Volume fraction based two-component model with $\sum_{i=1}^{m} \alpha^i = 1$, that defines mixture quantities as

$$\rho = \sum_{i=1}^{m} \alpha^i \rho^i\,, \quad \rho u_n = \sum_{i=1}^{m} \alpha^i \rho^i u_n^i\,, \quad \rho e = \sum_{i=1}^{m} \alpha^i \rho^i e^i$$

Assuming total pressure $p = (\gamma - 1)\,\rho e - \gamma p_\infty$ and speed of sound $c = (\gamma\,(p + p_\infty)/\rho)^{1/2}$ yields

$$\frac{p}{\gamma - 1} = \sum_{i=1}^{m} \frac{\alpha^i p^i}{\gamma^i - 1}\,, \quad \frac{\gamma p_\infty}{\gamma - 1} = \sum_{i=1}^{m} \frac{\alpha^i \gamma^i p_\infty^i}{\gamma^i - 1}$$

and the overall set of equations [Shyue, 1998]

$$\partial_t \rho + \partial_{x_n}(\rho u_n) = 0\,, \quad \partial_t(\rho u_k) + \partial_{x_n}(\rho u_k u_n + \delta_{kn} p) = 0\,, \quad \partial_t(\rho E) + \partial_{x_n}(u_n(\rho E + p)) = 0$$

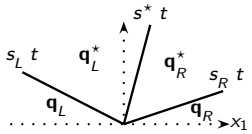
$$\frac{\partial}{\partial t}\left(\frac{1}{\gamma - 1}\right) + u_n \frac{\partial}{\partial x_n}\left(\frac{1}{\gamma - 1}\right) = 0\,, \quad \frac{\partial}{\partial t}\left(\frac{\gamma p_\infty}{\gamma - 1}\right) + u_n \frac{\partial}{\partial x_n}\left(\frac{\gamma p_\infty}{\gamma - 1}\right) = 0$$

Oscillation free at contacts: [Abgrall and Karni, 2001][Shyue, 2006]

Approximate Riemann solver

Use HLLC approach because of robustness and positivity preservation

$$\mathbf{q}^{HLLC}(x_1, t) = \begin{cases} \mathbf{q}_L\,, & x_1 < s_L\,t, \\ \mathbf{q}_L^\star\,, & s_L\,t \le x_1 < s^\star\,t, \\ \mathbf{q}_R^\star\,, & s^\star\,t \le x_1 \le s_R\,t, \\ \mathbf{q}_R\,, & x_1 > s_R\,t, \end{cases}$$



Wave speed estimates [Davis, 1988] $s_L = \min\{u_{1,L} - c_L, u_{1,R} - c_R\}$, $s_R = \max\{u_{1,L} + c_L, u_{1,R} + c_R\}$
Unkown state [Toro et al., 1994]

$$s^\star = \frac{p_R - p_L + s_L u_{1,L}(s_L - u_{1,L}) - \rho_R u_{1,R}(s_R - u_{1,R})}{\rho_L(s_L - u_{1,L}) - \rho_R(s_R - u_{1,R})}$$

$$\mathbf{q}_\tau^\star = \left[\eta, \eta s^\star, \eta u_2, \eta\left[\frac{(\rho E)_\tau}{\rho_\tau} + (s^\star - u_{1,\tau})\left(s_\tau + \frac{p_\tau}{\rho_\tau(s_\tau - u_{1,\tau})}\right)\right], \frac{1}{\gamma_\tau - 1}, \frac{\gamma_\tau p_{\infty,\tau}}{\gamma_\tau - 1}\right]^T$$

$$\eta = \rho_\tau \frac{s_\tau - u_{1,\tau}}{s_\tau - s^\star}\,, \quad \tau = \{L, R\}$$

Evaluate waves as $\mathcal{W}_1 = \mathbf{q}_L^\star - \mathbf{q}_L$, $\mathcal{W}_2 = \mathbf{q}_R^\star - \mathbf{q}_L^\star$, $\mathcal{W}_3 = \mathbf{q}_R - \mathbf{q}_R^\star$ and $\lambda_1 = s_L$, $\lambda_2 = s^\star$, $\lambda_3 = s_R$ to compute the fluctuations $\mathcal{A}^-\Delta = \sum_{\lambda_\nu < 0} \lambda_\nu \mathcal{W}_\nu$, $\mathcal{A}^+\Delta = \sum_{\lambda_\nu \ge 0} \lambda_\nu \mathcal{W}_\nu$ for $\nu = \{1, 2, 3\}$

Overall scheme: Wave Propagation method [Shyue, 2006]

Underwater explosion FSI simulations

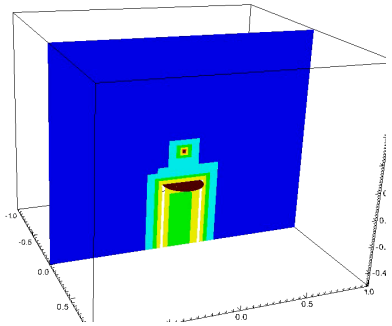
- Air: $\gamma^A = 1.4$, $p_\infty^A = 0$, $\rho^A = 1.29\,\mathrm{kg/m^3}$
- Water: $\gamma^W = 7.415$, $p_\infty^W = 296.2\,\mathrm{MPa}$, $\rho^W = 1027\,\mathrm{kg/m^3}$
- Cavitation modeling with pressure cut-off model at $p = -1\,\mathrm{MPa}$
- 3D simulation of deformation of air backed aluminum plate with $r = 85\,\mathrm{mm}$, $h = 3\,\mathrm{mm}$ from underwater explosion
  - Water basin [Ashani and Ghamsari, 2008] $2\,\mathrm{m} \times 1.6\,\mathrm{m} \times 2\,\mathrm{m}$
  - Explosion modeled as energy increase ($m_{C4} \cdot 6.06\,\mathrm{MJ/kg}$) in sphere with r=5mm
  - $\rho_s = 2719\,\mathrm{kg/m3}$, $E = 69\,\mathrm{GPa}$, $\nu = 0.33$, J2 plasticity model, yield stress $\sigma_y = 217.6\,\mathrm{MPa}$
- 3D simulation of copper plate $r = 32\,\mathrm{mm}$, $h = 0.25\,\mathrm{mm}$ rupturing due to water hammer
  - Water-filled shocktube $1.3\,\mathrm{m}$ with driver piston [Deshpande et al., 2006]
  - Piston simulated with separate level set, see [Deiterding et al., 2009] for pressure wave
  - $\rho_s = 8920\,\mathrm{kg/m3}$, $E = 130\,\mathrm{GPa}$, $\nu = 0.31$, J2 plasticity model, $\sigma_y = 38.5\,\mathrm{MPa}$, cohesive interface model, max. tensile stress $\sigma_c = 525\,\mathrm{MPa}$

Underwater explosion simulation

- AMR base grid $50 \times 40 \times 50$, $r_{1,2,3} = 2$, $r_4 = 4$, $l_c = 3$, highest level restricted to initial explosion center, 3rd and 4th level to plate vicinity
- Triangular mesh with 8148 elements
- Computations of 1296 coupled time steps to $t_{end} = 1\,\mathrm{ms}$
- 10+2 nodes 3.4 GHz Intel Xeon dual processor, $\sim 130\,\mathrm{h}$ CPU

Maximal deflection [mm]

	Exp.	Sim.
$20\,\mathrm{g}$, $d = 25\,\mathrm{cm}$	28.83	25.88
$30\,\mathrm{g}$, $d = 30\,\mathrm{cm}$	30.09	27.31



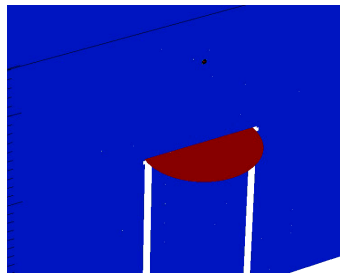
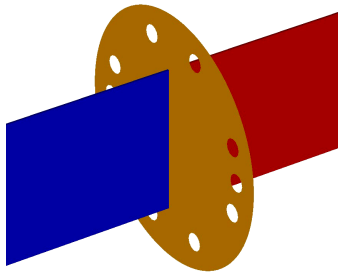
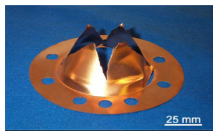


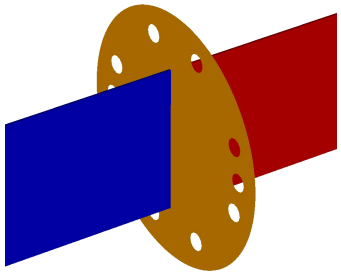
Plate in underwater shocktube

- AMR base mesh $374 \times 20 \times 20$, $r_{1,2} = 2$, $l_c = 2$, solid mesh: 8896 triangles
- ~ 1250 coupled time steps to $t_{end} = 1\,\mathrm{ms}$
- 6+6 nodes 3.4 GHz Intel Xeon dual processor, $\sim 800\,\mathrm{h}$ CPU





$p_0 = 64\,\mathrm{MPa}$



$p_0 = 173\,\mathrm{MPa}$

Outline

Favre-averaged Navier-Stokes equations

$$\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial}{\partial x_n}(\bar{\rho}\tilde{u}_n) = 0$$

$$\frac{\partial}{\partial t}(\bar{\rho}\tilde{u}_k) + \frac{\partial}{\partial x_n}(\bar{\rho}\tilde{u}_k\tilde{u}_n + \delta_{kn}\bar{p} - \tilde{\tau}_{kn} + \sigma_{kn}) = 0$$

$$\frac{\partial \bar{\rho}\bar{E}}{\partial t} + \frac{\partial}{\partial x_n}(\tilde{u}_n(\bar{\rho}\bar{E} + \bar{p}) + \tilde{q}_n - \tilde{\tau}_{nj}\tilde{u}_j + \sigma^e_n) = 0$$

$$\frac{\partial}{\partial t}(\bar{\rho}\tilde{Y}_i) + \frac{\partial}{\partial x_n}(\bar{\rho}\tilde{Y}_i\tilde{u}_n + \tilde{J}^i_n + \sigma^i_n) = 0$$

with stress tensor

$$\tilde{\tau}_{kn} = \tilde{\mu}\left(\frac{\partial \tilde{u}_n}{\partial x_k} + \frac{\partial \tilde{u}_k}{\partial x_n}\right) - \frac{2}{3}\tilde{\mu}\frac{\partial \tilde{u}_j}{\partial x_j}\delta_{in}\,,$$

heat conduction

$$\tilde{q}_n = -\tilde{\lambda}\frac{\partial \tilde{T}}{\partial x_n}\,,$$

and inter-species diffusion

$$\tilde{J}^i_n = -\bar{\rho}\tilde{D}_i\frac{\partial \tilde{Y}_i}{\partial x_n}$$

Favre-filtering

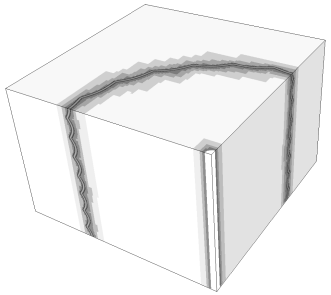
$$\tilde{\phi} = \frac{\overline{\rho\phi}}{\bar{\rho}} \quad \text{with} \quad \bar{\phi}(\mathbf{x}, t; \Delta_c) = \int_\Omega G(\mathbf{x} - \mathbf{x}'; \Delta_c)\phi(\mathbf{x}', t)d\mathbf{x}'$$

Numerical solution approach

- Subgrid terms σ_{kn}, σ^e_n, σ^i_n are computed by Pullin's stretched-vortex model
- Cutoff Δ_c is set to local SAMR resolution Δx_l
- It remains to solve the Navier-Stokes equations in the hyperbolic regime
  - 3rd order WENO method (hybridized with a tuned centered difference stencil) for convection
  - 2nd order conservative centered differences for diffusion

Example: Cylindrical Richtmyer-Meshkov instability

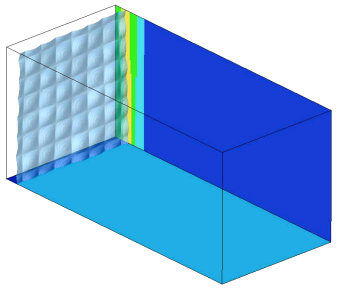
- Sinusoidal interface between two gases hit by shock wave
- Objective is correctly predict turbulent mixing
- Embedded boundary method used to regularize apex
- AMR base grid $95 \times 95 \times 64$ cells, $r_{1,2,3} = 2$
- $\sim 70,000\,\mathrm{h}$ CPU on 32 AMD 2.5GHZ-quad-core nodes

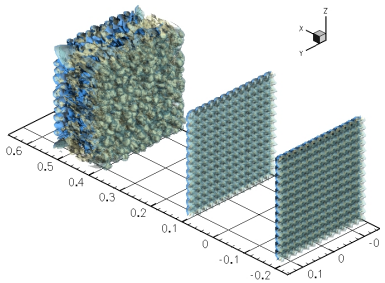


Planar Richtmyer-Meshkov instability

- Perturbed Air-SF6 interface shocked and re-shocked by Mach 1.5 shock
- Containment of turbulence in refined zones
- 96 CPUs IBM SP2-Power3
- WENO-TCD scheme with LES model
- AMR base grid $172 \times 56 \times 56$, $r_{1,2} = 2$, $10\,\mathrm{M}$ cells in average instead of $3\,\mathrm{M}$ (uniform)

Task	2ms (%)	5ms (%)	10ms (%)
Integration	45.3	65.9	52.0
Boundary setting	44.3	28.6	41.9
Flux correction	7.2	3.4	4.1
Interpolation	0.9	0.4	0.3
Reorganization	1.6	1.2	1.2
Misc.	0.6	0.5	0.5
Max. imbalance	1.25	1.23	1.30





Flux correction for Runge-Kutta method

Recall Runge-Kutta temporal update

$$\tilde{\mathbf{Q}}^\upsilon_j = \alpha_\upsilon \mathbf{Q}^n_j + \beta_\upsilon \tilde{\mathbf{Q}}^{\upsilon-1}_j + \gamma_\upsilon \frac{\Delta t}{\Delta x_k}\Delta \mathbf{F}^k(\tilde{\mathbf{Q}}^{\upsilon-1})$$

rewrite scheme as

$$\mathbf{Q}^{n+1} = \mathbf{Q}^n - \sum_{\upsilon=1}^{\Upsilon}\varphi_\upsilon \frac{\Delta t}{\Delta x_k}\Delta \mathbf{F}^k(\tilde{\mathbf{Q}}^{\upsilon-1}) \quad \text{with} \quad \varphi_\upsilon = \gamma_\upsilon \prod_{\nu=\upsilon+1}^{\Upsilon}\beta_\nu$$

Flux correction to be used [Pantano et al., 2007]

1. $\delta \mathbf{F}^{1,l+1}_{i-\frac{1}{2},j} := -\varphi_1 \mathbf{F}^{1,l}_{i-\frac{1}{2},j}(\tilde{\mathbf{Q}}^0)\,, \qquad \delta \mathbf{F}^{1,l+1}_{i-\frac{1}{2},j} := \delta \mathbf{F}^{1,l+1}_{i-\frac{1}{2},j} - \sum_{\upsilon=2}^{\Upsilon}\varphi_\upsilon \mathbf{F}^{1,l}_{i-\frac{1}{2},j}(\tilde{\mathbf{Q}}^{\upsilon-1})$
2. $\delta \mathbf{F}^{1,l+1}_{i-\frac{1}{2},j} := \delta \mathbf{F}^{1,l+1}_{i-\frac{1}{2},j} + \frac{1}{r^2_{l+1}}\sum_{m=0}^{r_{l+1}-1}\sum_{\upsilon=1}^{\Upsilon}\varphi_\upsilon \mathbf{F}^{1,l+1}_{v+\frac{1}{2},w+m}\left(\tilde{\mathbf{Q}}^{\upsilon-1}(t + \kappa\Delta t_{l+1})\right)$

Storage-efficient SSPRK(3,3):

υ	α_υ	β_υ	γ_υ	φ_υ
1	1	0	1	$\frac{1}{6}$
2	$\frac{3}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{6}$
3	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$

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Lecture 4a

Using the SAMR approach for elliptic and parabolic problems

Course *Block-structured Adaptive Mesh Refinement Methods for Conservation Laws*
Theory, Implementation and Application

Ralf Deiterding
Computer Science and Mathematics Division
Oak Ridge National Laboratory
P.O. Box 2008 MS6367, Oak Ridge, TN 37831, USA

E-mail: deiterdingr@ornl.gov

Outline

Poisson equation

$$\begin{aligned} \Delta q(\mathbf{x}) &= \psi(\mathbf{x}), \quad \mathbf{x} \in \Omega \subset \mathbb{R}^d, \quad q \in \mathrm{C}^2(\Omega), \quad \psi \in \mathrm{C}^0(\Omega) \\ q &= \psi^\Gamma(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega \end{aligned}$$

Discrete Poisson equation in 2D:

$$\frac{Q_{j+1,k} - 2Q_{jk} + Q_{j-1,k}}{\Delta x_1^2} + \frac{Q_{j,k+1} - 2Q_{jk} + Q_{j,k-1}}{\Delta x_2^2} = \psi_{jk}$$

Operator

$$\mathcal{A}(Q_{\Delta x_1, \Delta x_2}) = \begin{bmatrix} & \frac{1}{\Delta x_2^2} & \\ \frac{1}{\Delta x_1^2} & -\left(\frac{2}{\Delta x_1^2} + \frac{2}{\Delta x_2^2}\right) & \frac{1}{\Delta x_2^2} \\ & \frac{1}{\Delta x_2^2} & \end{bmatrix} Q(x_{1,j}, x_{2,k}) = \psi_{jk}$$

$$Q_{jk} = \frac{1}{\sigma}\left[(Q_{j+1,k} + Q_{j-1,k})\Delta x_2^2 + (Q_{j,k+1} + Q_{j,k-1})\Delta x_1^2 - \Delta x_1^2 \Delta x_2^2 \psi_{jk}\right]$$

with $\sigma = \dfrac{2\Delta x_1^2 + 2\Delta x_2^2}{\Delta x_1^2 \Delta x_2^2}$

Iterative methods

Jacobi iteration

$$Q_{jk}^{m+1} = \frac{1}{\sigma}\left[(Q_{j+1,k}^m + Q_{j-1,k}^m)\Delta x_2^2 + (Q_{j,k+1}^m + Q_{j,k-1}^m)\Delta x_1^2 - \Delta x_1^2 \Delta x_2^2 \psi_{jk}\right]$$

Lexicographical Gauss-Seidel iteration (use updated values when they become available)

$$Q_{jk}^{m+1} = \frac{1}{\sigma}\left[(Q_{j+1,k}^m + Q_{j-1,k}^{m+1})\Delta x_2^2 + (Q_{j,k+1}^m + Q_{j,k-1}^{m+1})\Delta x_1^2 - \Delta x_1^2 \Delta x_2^2 \psi_{jk}\right]$$

Efficient parallelization / patch-wise application not possible!

Checker-board or Red-Black Gauss Seidel iteration

1. $Q_{jk}^{m+1} = \frac{1}{\sigma}\left[(Q_{j+1,k}^m + Q_{j-1,k}^m)\Delta x_2^2 + (Q_{j,k+1}^m + Q_{j,k-1}^m)\Delta x_1^2 - \Delta x_1^2 \Delta x_2^2 \psi_{jk}\right]$ for $j + k \bmod 2 = 0$
2. $Q_{jk}^{m+1} = \frac{1}{\sigma}\left[(Q_{j+1,k}^{m+1} + Q_{j-1,k}^{m+1})\Delta x_2^2 + (Q_{j,k+1}^{m+1} + Q_{j,k-1}^{m+1})\Delta x_1^2 - \Delta x_1^2 \Delta x_2^2 \psi_{jk}\right]$ for $j + k \bmod 2 = 1$

Gauss-Seidel methods require $\sim 1/2$ of iterations than Jacobi method, however, iteration count still proportional to number of unknowns [Hackbusch, 1994]

Smoothing vs. solving

ν iterations with iterative linear solver

$$Q^{m+\nu} = \mathcal{S}(Q^m, \psi, \nu)$$

Defect after m iterations

$$d^m = \psi - \mathcal{A}(Q^m)$$

Defect after $m + \nu$ iterations

$$d^{m+\nu} = \psi - \mathcal{A}(Q^{m+\nu}) = \psi - \mathcal{A}(Q^m + v_\nu^m) = d^m - \mathcal{A}(v_\nu^m)$$

with correction

$$v_\nu^m = \mathcal{S}(\vec{0}, d^m, \nu)$$

Neglecting the sub-iterations in the smoother we write

$$Q^{n+1} = Q^n + v = Q^n + \mathcal{S}(d^n)$$

Observation: Oscillations are damped faster on coarser grid.

Coarse grid correction:

$$Q^{n+1} = Q^n + v = Q^n + \mathcal{PSR}(d^n)$$

where $\mathcal{R}$ is suitable restriction operator and $\mathcal{P}$ a suitable prolongation operator

Two-grid correction method

Relaxation on current grid:

$$\bar{Q} = \mathcal{S}(Q^n, \psi, \nu)$$

$$Q^{n+1} = \bar{Q} + \mathcal{PS}(\vec{0}, \cdot, \mu)\mathcal{R}(\psi - \mathcal{A}(\bar{Q}))$$

Algorithm:

$\bar{Q} := \mathcal{S}(Q^n, \psi, \nu)$
$d := \psi - \mathcal{A}(\bar{Q})$

$d_c := \mathcal{R}(d)$
$v_c := \mathcal{S}(0, d_c, \mu)$
$v := \mathcal{P}(v_c)$
$Q^{n+1} := \bar{Q} + v$

with smoothing:

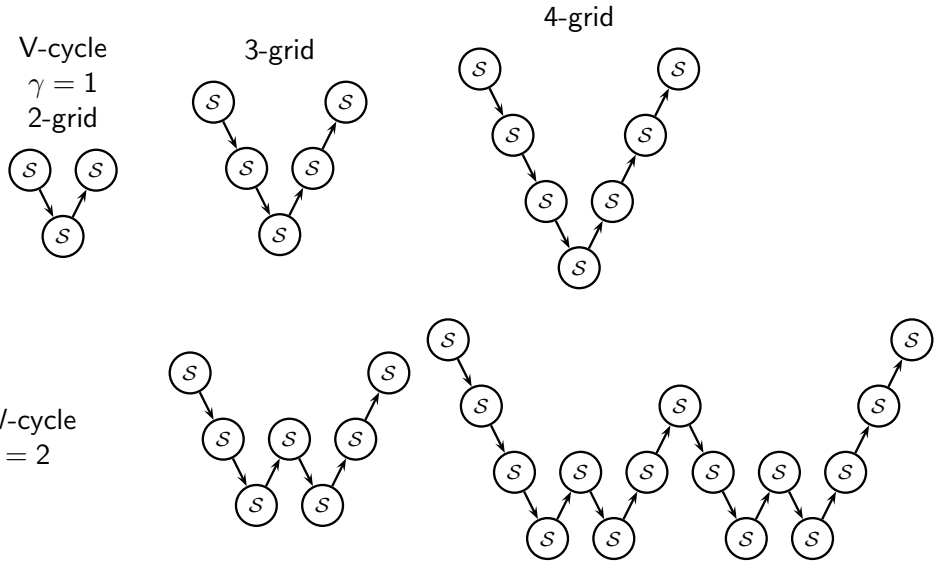
$d := \psi - \mathcal{A}(Q)$
$v := \mathcal{S}(0, d, \nu)$
$r := d - \mathcal{A}(v)$
$d_c := \mathcal{R}(r)$
$v_c := \mathcal{S}(0, d_c, \mu)$
$v := v + \mathcal{P}(v_c)$
$Q^{n+1} := Q + v$

with pre- and post-iteration:

$d := \psi - \mathcal{A}(Q)$
$v := \mathcal{S}(0, d, \nu_1)$
$r := d - \mathcal{A}(v)$
$d_c := \mathcal{R}(r)$
$v_c := \mathcal{S}(0, d_c, \mu)$
$v := v + \mathcal{P}(v_c)$
$d := d - \mathcal{A}(v)$
$r := \mathcal{S}(0, d, \nu_2)$
$Q^{n+1} := Q + v + r$

[Hackbusch, 1985]

Multi-level methods and cycles



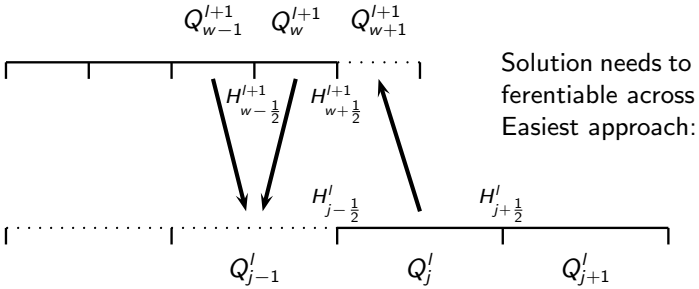
[Hackbusch, 1985] [Wesseling, 1992] ...

Stencil modification at coarse-fine boundaries in 1D

1D Example: Cell j, $\psi - \nabla \cdot \nabla q = 0$

$$d_j^l = \psi_j - \frac{1}{\Delta x_l}\left(\frac{1}{\Delta x_l}(Q_{j+1}^l - Q_j^l) - \frac{1}{\Delta x_l}(Q_j^l - Q_{j-1}^l)\right) = \psi_j - \frac{1}{\Delta x_l}\left(H_{j+\frac{1}{2}}^l - H_{j-\frac{1}{2}}^l\right)$$

H is approximation to *derivative* of Q^l.
Consider 2-level situation with $r_{l+1} = 2$:



Solution needs to be continuously differentiable across interface.
Easiest approach: $H_{w+\frac{1}{2}}^{l+1} \equiv H_{j-\frac{1}{2}}^l$

No specific modification necessary for 1D vertex-based stencils, cf. [Bastian, 1996]

Stencil modification at coarse-fine boundaries in 1D II

Set $H_{w+\frac{1}{2}}^{l+1} = H_{\mathcal{I}}$. Inserting Q gives

$$\frac{Q_{w+1}^{l+1} - Q_w^{l+1}}{\Delta x_{l+1}} = \frac{Q_j^l - Q_w^{l+1}}{\frac{3}{2}\Delta x_{l+1}}$$

from which we readily derive

$$Q_{w+1}^{l+1} = \frac{2}{3}Q_j^l + \frac{1}{3}Q_w^{l+1}$$

for the boundary cell on $l+1$. We use the flux correction procedure to enforce $H_{w+\frac{1}{2}}^{l+1} \equiv H_{j-\frac{1}{2}}^l$ and thereby $H_{j-\frac{1}{2}}^l \equiv H_{\mathcal{I}}$.

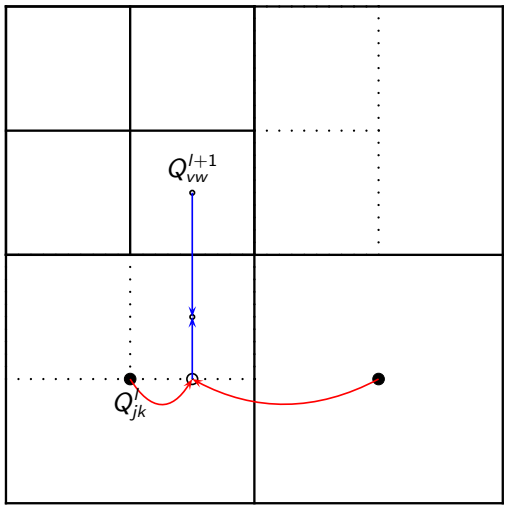
Correction pass [Martin, 1998]

1. $\delta H_{j-\frac{1}{2}}^{l+1} := -H_{j-\frac{1}{2}}^l$
2. $\delta H_{j-\frac{1}{2}}^{l+1} := \delta H_{j-\frac{1}{2}}^{l+1} + H_{w+\frac{1}{2}}^{l+1} = -H_{j-\frac{1}{2}}^l + (Q_j^l - Q_w^{l+1})/\frac{3}{2}\Delta x_{l+1}$
3. $\check{d}_j^l := d_j^l + \frac{1}{\Delta x_l}\delta H_{j-\frac{1}{2}}^{l+1}$

yields

$$\check{d}_j^l = \psi_j - \frac{1}{\Delta x_l}\left(\frac{1}{\Delta x_l}(Q_{j+1}^l - Q_j^l) - \frac{2}{3\Delta x_{l+1}}(Q_j^l - Q_w^{l+1})\right)$$

Stencil modification at coarse-fine boundaries: 2D



$$Q_{v,w-1}^{l+1} = \frac{1}{3}Q_{vw}^{l+1} + \frac{2}{3}\left(\frac{3}{4}Q_{jk}^l + \frac{1}{4}Q_{j+1,k}^l\right)$$

In general:

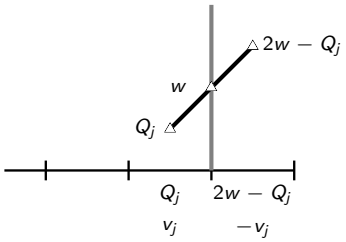
$$Q_{v,w-1}^{l+1} = \left(1 - \frac{2}{r_{l+1}+1}\right)Q_{vw}^{l+1} + \frac{2}{r_{l+1}+1}\left((1-f)Q_{jk}^l + fQ_{j+1,k}^l\right)$$

with

$$f = \frac{x_{1,l+1}^v - x_{1,l}^j}{\Delta x_{1,l}}$$

Components of an SAMR multigrid method

- Stencil operators
  - Application of defect $d^l = \psi^l - \mathcal{A}(Q^l)$ on each grid $G_{l,m}$ of level l
  - Computation of correction $v^l = \mathcal{S}(0, d^l, \nu)$ on each grid of level l
- Boundary (ghost cell) operators
  - Synchronization of Q^l and v^l on $\tilde{S}_l^1$
  - Specification of Dirichlet boundary conditions for a finite volume discretization for $Q^l \equiv w$ and $v^l \equiv w$ on $\tilde{P}_l^1$
  - Specification of $v^l \equiv 0$ on $\tilde{I}_l^1$
  - Specification of $Q_l = \frac{(r_l-1)Q^{l+1}+2Q^l}{r_l+1}$ on $\tilde{I}_l^1$



- Coarse-fine boundary flux accumulation and application of δH^{l+1} on defect d^l
- Standard prolongation and restriction on grids between adjacent levels
- Adaptation criteria
  - E.g., standard restriction to project solution on 2x coarsended grid, then use local error estimation
- Looping instead of time steps and check of convergence

Additive geometric multigrid algorithm

`AdvanceLevelMG(l) - Correction Scheme`

Set ghost cells of Q^l	
Calculate defect d^l from Q^l, ψ^l	$d^l := \psi^l - \mathcal{A}(Q^l)$
If $(l < l_{max})$	
Calculate updated defect r^{l+1} from v^{l+1}, d^{l+1}	$r^{l+1} := d^{l+1} - \mathcal{A}(v^{l+1})$
Restrict d^{l+1} onto d^l	$d^l := \mathcal{R}_l^{l+1}(r^{l+1})$
Do ν_1 smoothing steps to get correction v^l	$v^l := \mathcal{S}(0, d^l, \nu_1)$
If $(l > l_{min})$	
Do $\gamma > 1$ times	
AdvanceLevelMG$(l-1)$	
Set ghost cells of v^{l-1}	
Prolongate and add v^{l-1} to v^l	$v^l := v^l + \mathcal{P}_l^{l-1}(v^{l-1})$
If $(\nu_2 > 0)$	
Set ghost cells of v^l	
Update defect d^l according to v^l	$d^l := d^l - \mathcal{A}(v^l)$
Do ν_2 post-smoothing steps to get r^l	$r^l := \mathcal{S}(v^l, d^l, \nu_2)$
Add addional correction r^l to v^l	$v^l := v^l + r^l$
Add correction v^l to Q^l	$Q^l := Q^l + v^l$

Additive Geometric Multiplicative Multigrid Algorithm

`Start - Start iteration on level` l_{max}

```
For l = l_max Downto l_min + 1 Do
      Restrict Q^l onto Q^{l-1}
Regrid(0)
AdvanceLevelMG(l_max)
```

See also: [Trottenberg et al., 2001], [Canu and Ritzdorf, 1994]
Vertex-based: [Brandt, 1977], [Briggs et al., 2001]

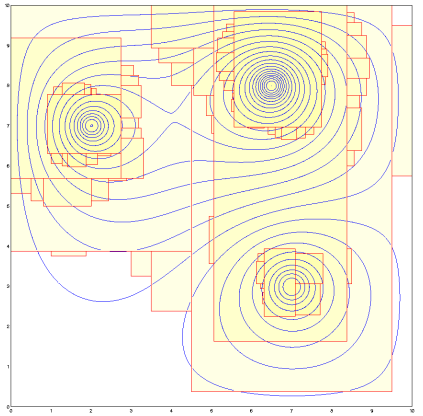
Example

On $\Omega = [0, 10] \times [0, 10]$ use hat function

$$\psi = \begin{cases} -A_n \cos\left(\frac{\pi r}{2R_n}\right), & r < R_n \\ 0 & \text{elsewhere} \end{cases}$$

with $r = \sqrt{(x_1 - X_n)^2 + (x_2 - Y_n)^2}$ to define three sources with

n	A_n	R_n	X_n	Y_n
1	0.3	0.3	6.5	8.0
2	0.2	0.3	2.0	7.0
3	-0.1	0.4	7.0	3.0



	128×128	1024×1024	1024×1024
l_{max}	3	0	0
l_{min}	-4	-7	-4
ν_1	5	5	5
ν_2	5	5	5
V-Cycles	15	16	341
Time [sec]	9.4	27.7	563

Stop at $\|d^l\|_{max} < 10^{-7}$ for $l \geq 0$, $\gamma = 1$, $r_l = 2$

Some comments on parabolic problems

- Consequences of time step refinement
- Level-wise elliptic solves vs. global solve
- If time step refinement is used an elliptic flux correction is unavoidable.
- The correction is explained in Bell, J. (2004). Block-structured adaptive mesh refinement. Lecture 2. Available at https://ccse.lbl.gov/people/jbb/shortcourse/lecture2.pdf.

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Lecture 4b

Design of SAMR Systems, Advanced Parallelization, Usage

Course *Block-structured Adaptive Mesh Refinement Methods for Conservation Laws*
Theory, Implementation and Application

Ralf Deiterding
Computer Science and Mathematics Division
Oak Ridge National Laboratory
P.O. Box 2008 MS6367, Oak Ridge, TN 37831, USA

E-mail: deiterdingr@ornl.gov

Outline

Simplified structured designs

Distributed memory parallelization fully supported if not otherwise states.

- PARAMESH (Parallel Adaptive Mesh Refinement)
  - Library based on uniform refinement blocks [MacNeice et al., 2000]
  - Both multigrid and explicit algorithms considered
  - http://sourceforge.net/projects/paramesh
- Flash code (AMR code for astrophysical thermonuclear flashes)
  - Built on PARAMESH
  - Solves the magneto-hydrodynamic equations with self-gravitation
  - http://flash.uchicago.edu/website/home
- Uintah (AMR code for simulation of accidental fires and explosions)
  - Only explicit algorithms considered
  - FSI coupling Material Point Method and ICE Method (Implicit, Continuous fluid, Eulerian)
  - http://www.uintah.utah.edu
- DAGH/Grace [Parashar and Browne, 1997]
  - Just C++ data structures but no methods
  - All grids are aligned to bases mesh coarsened by factor 2
  - http://userweb.cs.utexas.edu/users/dagh

Systems that support general SAMR

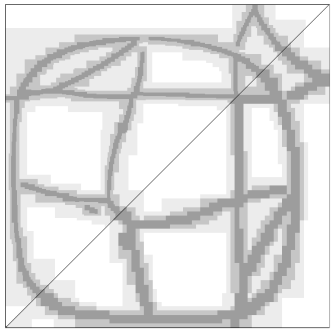
- SAMRAI - Structured Adaptive Mesh Refinement Application Infrastructure
  - Very mature SAMR system [Hornung et al., 2006]
  - Explicit algorithms directly supported, implicit methods through interface to Hypre package
  - Mapped geometry and some embedded boundary support
  - https://computation.llnl.gov/casc/SAMRAI
- BoxLib, AmrLib, MGLib, HGProj
  - Berkley-Lab-AMR collection of C++ classes by J. Bell et al., 50,000 LOC [Rendleman et al., 2000]
  - Both multigrid and explicit algorithms supported
  - https://ccse.lbl.gov/Software (no codes available)
- Chombo
  - Redesign and extension of BoxLib by P. Colella et al.
  - Both multigrid and explicit algorithms demonstrated
  - Some embedded boundary support
  - https://seesar.lbl.gov/anag/chombo

Further SAMR software

- Overture (Object-oriented tools for solving PDEs in complex geometries)
  - Overlapping meshes for complex geometries by W. Henshaw et al. [Brown et al., 1997]
  - Explicit and implicit algorithms supported
  - https://computation.llnl.gov/casc/Overture
- AMRClaw within Clawpack [Berger and LeVeque, 1998]
  - Serial 2D Fortran 77 code for the explicit Wave Propagation method with own memory management
  - http://www.clawpack.org
- Amrita by J. Quirk
  - Only 2D explicit finite volume methods supported
  - Embedded boundary algorithm
  - http://www.amrita-cfd.org
- Cell-based Cartesian AMR: RAGE
  - Embedded boundary method
  - Explicit and implicit algorithms
  - [Gittings et al., 2008]

AMROC

- "Adaptive Mesh Refinement in Object-oriented C++"
- $\sim 46,000$ LOC for C++ SAMR kernel, $\sim 140,000$ total C++, C, Fortran-77
- uses parallel hierarchical data structures that have evolved from DAGH
- Right: point explosion in box, 4 level, Euler computation, 7 compute nodes
- V1.0: http://amroc.sourceforge.net



	l_{max}	Level 0	Level 1	Level 2	Level 3	Level 4
AMROC's DAGH grids/cells	1	43/22500	145/38696			
	2	42/22500	110/48708	283/83688		
	3	36/22500	78/54796	245/109476	582/165784	
	4	41/22500	88/56404	233/123756	476/220540	1017/294828
Original DAGH grids/cells	1	238/22500	125/41312			
	2	494/22500	435/48832	190/105216		
	3	695/22500	650/55088	462/133696	185/297984	
	4	875/22500	822/57296	677/149952	428/349184	196/897024

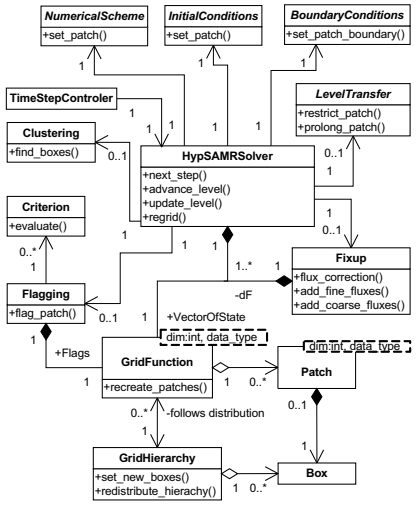
Comparison of number of cells and grids in DAGH and AMROC

The Virtual Test Facility

- Implements all described algorithms beside multigrid methods
- AMROC V2.0 plus solid mechanics solvers
- Implements explicit SAMR with different finite volume solvers
- Embedded boundary method, FSI coupling
- $\sim 430,000$ lines of code total in C++, C, Fortran-77, Fortran-90
- autoconf / automake environment with support for typical parallel high-performance system
- http://www.cacr.caltech.edu/asc
- [Deiterding et al., 2006][Deiterding et al., 2007]

UML design of AMROC

- Classical framework approach with generic main program in C++
- Customization / modification in Problem.h include file by derivation from base classes and redefining virtual interface functions
- Predefined, scheme-specific classes (with F77 interfaces) provided for standard simulations
- Standard simulations require only linking to F77 functions for initial and boundary conditions, source terms. No C++ knowledge required
- Interface mimics Clawpack
- Expert usage (algorithm modification, advanced output, etc.) in C++

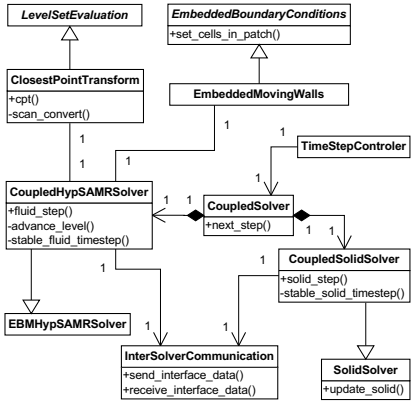


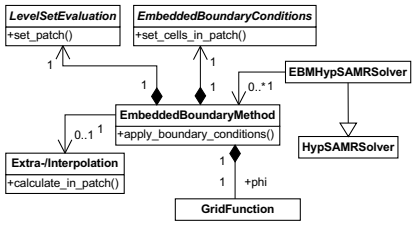
Commonalities in software design

- Index coordinate system based on $\Delta x_{n,l} \cong \prod_{\kappa=l+1}^{l_{max}} r_\kappa$ to uniquely identify a cell witin the hierarchy
- `Box<dim>`, `BoxList<dim>` class that define rectangular regions $G_{m,l}$ by `lowerleft, upperright, stepsize` and specify topological operations $\cap$, $\cup$, $\setminus$
- `Patch<dim,type>` class that assigns data to a rectangular grid $G_{m,l}$
- A class, here `GridFunction<dim,type>`, that defines topogical relations between lists of Patch objects to implement sychronization, restriction, prolongation, re-distribution
- Hierarchical parallel data structures are typically C++, routines on patches often Fortran

Embedded boundary method / FSI coupling

- Multiple independent EmbeddedBoundaryMethod objects possible
- Specialization of GFM boundary conditions, level set description in scheme-specific F77 interface classes





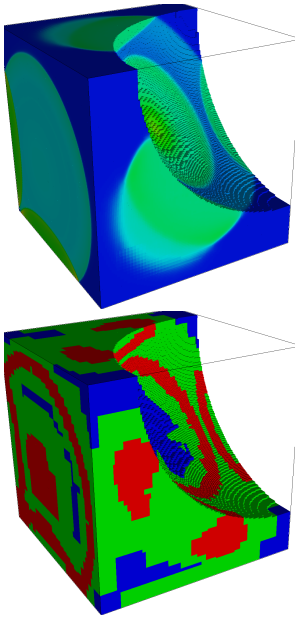
- Coupling algorithm implemented in further derived HypSAMRSolver class
- Level set evaluation always with CPT algorithm
- Parallel communication through efficient non-blocking communication module
- Time step selection for both solvers through CoupledSolver class

Performance assessment

- Test run on 2.2 GHz AMD Opteron quad-core cluster connected with Infiniband
- Cartesian test configuration
- Spherical blast wave, Euler equations, 3rd order WENO scheme, 3-step Runge-Kutta update
- AMR base grid 64^3, $r_{1,2} = 2$, 89 time steps on coarsest level
- With embedded boundary method: 96 time steps on coarsest level
- Redistribute in parallel every 2nd base level step
- Uniform grid $256^3 = 16.8 \cdot 10^6$ cells

Level	Grids	Cells
0	115	262,144
1	373	1,589,808
2	2282	5,907,064

Grid and cells used on 16 CPUs



Cost of SAMR (AMROC V2.0)

- Flux correction is negligible
- Clustering is negligible (already local approach). For the complexities of a scalable global clustering algorithm see [Gunney et al., 2007]
- Costs for GFM constant around $\sim 36\%$
- Main costs: `Regrid(1)` operation and ghost cell synchronization

CPUs	16	32	64
Time per step	32.44s	18.63s	11.87s
Uniform	59.65s	29.70s	15.15s
Integration	73.46%	64.69%	50.44%
Flux Correction	1.30%	1.49%	2.03%
Boundary Setting	13.72%	16.60%	20.44%
Regridding	10.43%	15.68%	24.25%
Clustering	0.34%	0.32%	0.26%
Output	0.29%	0.53%	0.92%
Misc.	0.46%	0.44%	0.47%

CPUs	16	32	64
Time per step	43.97s	25.24s	16.21s
Uniform	69.09s	35.94s	18.24s
Integration	59.09%	49.93%	40.20%
Flux Correction	0.82%	0.80%	1.14%
Boundary Setting	19.22%	25.58%	28.98%
Regridding	7.21%	9.15%	13.46%
Clustering	0.25%	0.23%	0.21%
GFM Find Cells	2.04%	1.73%	1.38%
GFM Interpolation	6.01%	10.39%	7.92%
GFM Overhead	0.54%	0.47%	0.37%
GFM Calculate	0.70%	0.60%	0.48%
Output	0.23%	0.52%	0.74%
Misc.	0.68%	0.62%	0.58%

AMROC scalability tests

Basic test configuration

- Spherical blast wave, Euler equations, 3D wave propagation method
- AMR base grid 32^3 with $r_{1,2} = 2, 4$. 5 time steps on coarsest level
- Uniform grid $256^3 = 16.8 \cdot 10^6$ cells, 19 time steps
- Flux correction deactivated
- No volume I/O operations
- Tests run IBM BG/P (mode VN)

Weak scalability test

- Reproduction of configuration each 64 CPUs
- On 1024 CPUs: $128 \times 64 \times 64$ base grid, $> 33,500$ Grids, $\sim 61 \cdot 10^6$ cells, uniform $1024 \times 512 \times 512 = 268 \cdot 10^6$ cells

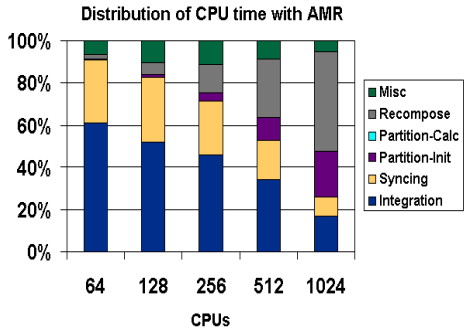
Level	Grids	Cells
0	606	32,768
1	575	135,312
2	910	3,639,040

Strong scalability test

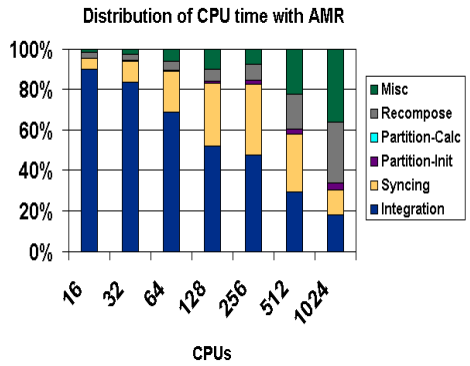
- $64 \times 32 \times 32$ base grid, uniform $512 \times 256 \times 256 = 33.6 \cdot 10^6$ cells

Level	Grids	Cells
0	1709	65,536
1	1735	271,048
2	2210	7,190,208

Scalability tests AMROC V2.0



weak scalability test



strong scalability test

- Syncing: Parallel communication portion of boundary setting
- Recompose: topological list operations, construction of boundary info, redistribution of data blocks
- Partition-Init, Partition-Calc: construction of space filling curve
- Costs for Partition-Init and Recompose increase dramatically for large problem size

Topological list operations

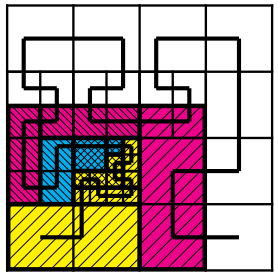
Weak scalability problems are due to non-parallel sub-algorithms!

- Operations $\cap$, $\setminus$ on two box lists have complexity $O(NM)$
- Costs of operations in `Recompose(l)`, that use global box lists G_l, increase quadratically, cf. [Wissink et al., 2003]
- Solution
  - Clip G_l with properly chosen quadratic bounding box around G_l^p before using $\cap$, $\setminus$
- All topological operations in `Recompose(l)` involving global lists can be reduced to local ones
- Present code V2.1β still uses `MPI_allgather()` to communicate global lists to all nodes
- Global view is particularly useful to evaluate new local portion of hierarchy and for data redistribution

Construction of space-filling curve

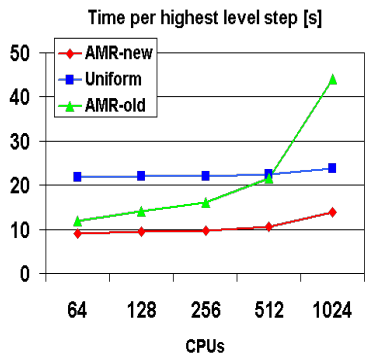
Computation of space filling curve

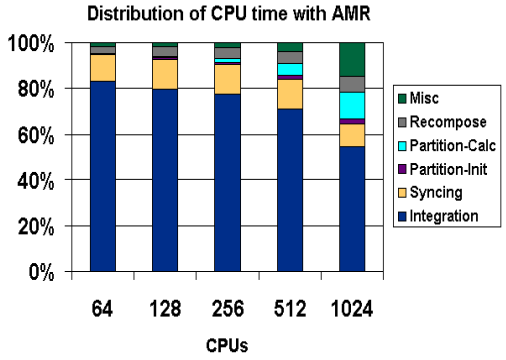
- Partition-Init
  1. Compute aggregated workload for new grid hierarchy G_l and project result onto level 0
  2. Construct recursively SFC-units until work in each unit is homogeneous, GuCFactor defines minimal coarseness relative to level-0 grid



- Partition-Calc
  1. Compute entire workload and new work for each processor
  2. Go sequentially through SFC-ordered list of partitioning units and assign units to processors, refine partition if necessary and possible
- Ensure scalability of Partition-Init by creating SFC-units strictly local
- Currently still use of `MPI_allgather()` to create globally identical input for Partition-Calc

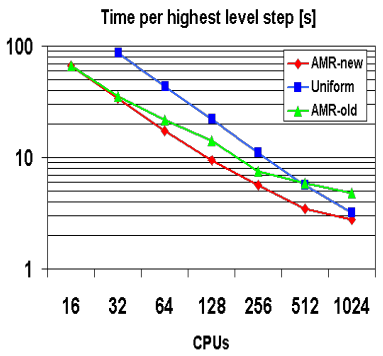
Weak scalability test V2.1β

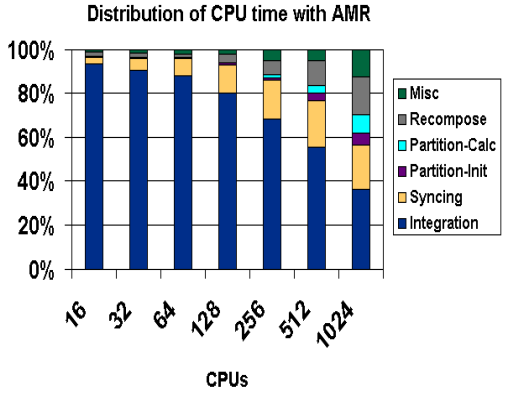




- Overall performance improvement for 1024 CPUs by $\sim 69\,\%$
- Absolute costs for Syncing are almost constant
- 1024 required usage of -DUAL option due to usage of global lists data structures in Partition-Calc and Recompose

Strong scalability test V2.1β





- Overall performance improvement for 1024 CPUs by 43 %
- Improved partitioning algorithm allowed usage of GuCFactor=1 instead of 2 before and full parallel data redistribution in every `Regrid(l)` instead of every 2nd level-0 step

To-Do

Comments

- Scalability of V2.1β for finite volume methods is comparable to results reported from Chombo and SAMRAI
- Significantly better scalability has so far only been reported for shallow hierarchies [Greenough et al., 2005] and/or tailored parameters choices

Next step

- Eliminate aggregation of global box list data (that currently uses simply `MPI_allgather()`)
  - Partition-Calc: assignment of SFC-ordered sequence and refinement could be executed sequentially on each node
  - Global topology lists: assemble only those portions of global lists on each node that are relevant for the subsequent operations. Use Cartesian bounding box information to construct irregular point-to-point communication pattern for list data between nodes

Future work

- Large-scale hierarchical I/O
- Hybrid parallelization (considering accelerators), cf. [Schive et al., 2010], [Jourdon, 2005]

Quick start with AMROC V2.0 I

Standard Linux development system assumed! See `install_vtf.pdf` for details.

1. Unpack `hdf4_src.tgz` into home directory: `cd; tar -xvzf hdf4_src.tgz`
2. `cd asc; ./build_hdf4.sh`
3. If last step successful `libdf.a`, `libjpeg.a`, `libmfhdf.a`, `libsz.a`, `libz.a` are in `$HOME/asc/hdf4/lib`.
4. Unpack `AMROC-Clawpack-1.0.tgz` into `$HOME/asc`: `cd $HOME/asc; tar -xvzf AMROC-Clawpack-1.0.tgz`
5. With `mpicc, mpicxx` commands
   - 5.1 `cd vtf`
   - 5.2 `./configure -C --enable-opt=yes --enable-mpi=yes HDF4_DIR=$HOME/asc/hdf4`
     If `autoconf, automake` are available add `--enable-maintainer-mode` to last line
   - 5.3 `cd gnu-opt-mpi`

Quick start with AMROC V2.0 II

   - 5.4 `make`
   - 5.5 `source ../ac/paths.sh`
   - 5.6 Optional unit test
     - 5.6.1 `../amroc/testrun.sh -m make -r 4 -s`
     - 5.6.2 `../amroc/testrun.sh -c`
6. Without MPI
   - 6.1 `cd vtf`
   - 6.2 `./configure -C --enable-opt=yes --enable-mpi=no HDF4_DIR=$HOME/asc/hdf4`
     If `autoconf, automake` are available add `--enable-maintainer-mode` to last line
   - 6.3 `cd gnu-opt`
   - 6.4 `make`
   - 6.5 `source ../ac/paths.sh`
   - 6.6 Optional unit test
     - 6.6.1 `../amroc/testrun.sh -m make -r 0 -s`
     - 6.6.2 `../amroc/testrun.sh -c`

Quick start with AMROC V2.0 III

7. Realistic example
   - 7.1 Change to compilation directory `gnu-opt/gnu-opt-mpi`: `cd amroc/clawpack/applications/euler/2d/SphereLiftOff`
   - 7.2 `make`
   - 7.3 Change to corresponding directory with solver.in: `cd vtf/amroc/clawpack/applications/euler/2d/SphereLiftOff`
   - 7.4 `./run.py` or `./run.py 2` (if you have compiled with MPI on a dual-core system)
   - 7.5 `gnuplot Density.gnu` shows density evolution of lower boundary
   - 7.6 Create binary VTK files for Paraview or VisIt: `hdf2tab.sh "-f display_file_visit.in"`
   - 7.7 Execute VisIt or Paraview and load the VTK files for visualization.

Quick start with AMROC V2.0 IV

8. FSI example (requires MPI)
   - 8.1 `cd $HOME/asc/gnu-opt-mpi/vtf/fsi/beam-amroc/VibratingBeam`
   - 8.2 `make`
   - 8.3 `cd $HOME/asc/vtf/fsi/beam-amroc/VibratingBeam`
   - 8.4 `./run.py 4`
     Change `LastNode` entry in `solver.in` for different processor number.
   - 8.5 `hdf2tab.sh`
   - 8.6 Execute VisIt or Paraview and load the VTK files for visualization.
9. For further documentation see http://www.cacr.caltech.edu/asc

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